Integral equations

An integral equation (IE) is an equation in which an unknown function occurs inside an integral. For example,

$$\int_a^b k(x,t)\phi(t)\,dt = f(x), \tag{73}$$

where $\phi(x)$ is the unknown function we wish to determine, while $f(x)$ and the so-called kernel $k(x,t)$ are given functions.

Any ODE problem plus boundary (or initial) conditions can be rewritten as an IE. For example, in the initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y(x)), \;\; y(a) = c,$$

integrating both sides from a to x yields the integral equation

$$y(x) = c + \int_a^x f(x, y(t))\,dt$$

How about a second order problem? For instance, consider

$$y'' + x^2 y = 0, \;\; y(a) = \alpha, \; y'(a) = \beta.$$

Integrating from a to x once gives

$$y'(x) - \alpha + \int_a^x t^2 y(t)\,dt = 0.$$

We can then integrate again, and eliminate the double integral by noticing that

$$\int_a^x \int_a^s f(t)\,dt\,ds = \int_a^x \int_t^x f(t)\,ds\,dt = \int_a^x f(t)(\int_t^x ds)\,dt = \int_a^x (x-t) f(t)\,dt$$

a simple trick of switching the order of integration in the triangular region. Thus, our ODE becomes the IE

$$y(x) = \beta + \alpha + \int_a^x t^2 y(t)(x-t)\, dt.$$

Fredholm Integral Equation

The IE's above came from initial value problems (IVP). IVPs always lead to integral equations called Volterra Integral Equations, characterised by the variable x as a limit in the integral.

If instead, we have a BVP on $a \le x \le b$ with boundary conditions at a and b, the tricks above won't work. But ODE BVPs can also be converted to IEs, using Green's functions. Consider the equation

$$Lu = r(x)u, \quad a \le x \le b, \tag{74}$$

with homogeneous boundary conditions at a and b, where L is a linear differential operator. If we were to find the Green's function for the operator, i.e. $g(x,\xi)$ such that

$$Lg = \delta(x-\xi)$$

then we could construct the solution to (74) as

$$u(x) = \int_a^b g(x,\xi) r(\xi) u(\xi)\, d\xi. \tag{75}$$

Integral equations such as (75), that come from BVPs, are called Fredholm Integral Equations. To summarize:

1. IVPs correspond to Volterra equations
2. BVPs correspond to Fredholm Integral Equations (FIE)
3. Boundary (or initial) conditions are implicitly included in the IE
4. The ODE and IE are equivalent, solving one is equivalent to solving the other

In this course, we will primarily focus on FIEs, of which there are two types:

1st kind FIE:

$$Ly = \int_a^b k(x,t)y(t)\ dt = f(x)$$

2nd kind FIE:

$$Ly = ry(x) + \int_a^b k(x,t)y(t)\ dt = f(x)$$

where the kernel function $k(x,t)$, $f(x)$, the constant r are all known, and $y(x)$ is to be determined.

Degenerate (separable) kernels

The difficulty of solving the integral equation depends on the complexity of the kernel. The simplest problems are those for which the kernel is made up of a finite sum of terms, each of which is a product of functions of x and functions of t, that is[8]

$$k(x,t) = \sum_{j=1}^{n} \alpha_j(x)\beta_j(t).$$

In this case the kernel is called *degenerate*, or separable. We will generally assume that the functions in each set $\{\alpha_j\}$ and $\{\beta_j\}$ are linearly independent (otherwise, the kernel can be rewritten in this form with a smaller n).

Example:

$$k(x,t) = 4xt - 5x^2t^2$$

[8]You might think of $k(x,t)$ as a finite series of "separation of variables" products.

$(n = 2,\ \alpha_1 = x_1,\ \alpha_2 = x^2,\ \beta_1 = 4t, \beta_2 = -5t^2)$

How do we find eigenvalues/eigenfunctions of $Ly = \lambda y$, in the case of a separable kernel?[9]
For the example given, we must solve

$$\int_0^1 k(x,t)y(t)\mathrm{d}t = \lambda y(x) \tag{76}$$

$$\Rightarrow \int_0^1 4xty\mathrm{d}t + \int_0^1 -5x^2t^2y\mathrm{d}t = \lambda y$$

The great thing about degenerate kernels is that the functions of x can be pulled out of the integral. What's left integrates to a constant, and thus we are left with a perfect visual of the requisite form of $y(x)$. In steps:

- Factor out $\alpha_j(x) \to \int \beta y\mathrm{d}t$ integrals

$$\underbrace{\left(\int_0^1 4ty\mathrm{d}t\right)}_{\text{constant}_1} x + \underbrace{\left(\int_0^1 -5t^2y\mathrm{d}t\right)}_{\text{constant}_2} x^2 = \lambda y(x)$$

- Determines form: $y(x) = C_1x + C_2x^2$. For $\lambda \neq 0$ all eigenfunctions will be linear combinations of $\alpha_j(x)$ functions.

- Plug in $y(t)$, evaluate integrals

$$\left(\frac{4}{3}C_1 + C_2\right)x + \left(-\frac{5}{4}C_1 - C_2\right)x^2 = \lambda C_1 x + \lambda C_2 x^2$$

- Equate coeffs

$$\begin{matrix} x: \\ x^2: \end{matrix} \quad \begin{pmatrix} \frac{4}{3} & 1 \\ -\frac{5}{4} & -1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \lambda \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$\to$ matrix eigenvalue problem

$$\begin{matrix} \lambda_1 = \frac{1}{2} & y_1(x) = -6x + 5x^2 \\ \lambda_2 = -\frac{1}{6} & y_2(x) = 2x - 3x^2 \end{matrix}$$

[9]You'll notice we are switching to writing $Ly = \lambda y$ now, instead of $Ly = -\lambda y$. Oddly enough, while BVPs often use the "$-\lambda$", integral equations almost never do. Lesson learned: the world is an inconsistent place.

Solving the algebraic system gives a finite set of λ_ks. Usually, we find one y_k for each $\lambda_k \to$ "multiplicity 1".

But what about zero eigenvalues? Is there a function y for which $Ly = 0$? Actually, there are infinite! To see this, note that $Ly = 0$ if

$$\int_0^1 \beta_j(t)y(t)\mathrm{d}t = 0 \qquad \text{for } j = 1, 2$$

that is if each separate integral evaluates to zero. Since the βs are polynomials of degree 1 and 2, we can "build" lots (∞) of functions orthogonal to both β_i by taking polynomials of the same form plus one non-matching degree[10]. That is, we look for $y_{0,k}(x)$ of the form

$$y_{0,k} = c_1 x + c_2 x^2 + x^k \qquad k = 0, 3, 4, \dots \tag{77}$$

Now, plug in and choose c_1, c_2 so that $y_{0,k}$ is orthogonal to β_1, β_2

$$\begin{aligned} \Rightarrow y_{0,1}(x) &= 4x + \frac{10}{3}x^2 + 1 \\ y_{0,2}(x) &= \frac{2}{5}x - \frac{4}{5}x^2 + x^3 \\ y_{0,3}(x) &= \dots \end{aligned}$$

We would say that $\lambda = 0$ is an eigenvalue of "multiplicity ∞."

Next we want to solve $Ly = f$. FAT will be a useful tool, but first we need to define the adjoint.

Adjoint for Integral operators

We use the same definition as for differential operators: $\langle Ly, w\rangle = \langle y, L^* w\rangle$, with inner product

$$\langle u, v\rangle = \int_a^b u(x)v(x)\mathrm{d}x$$

Let

$$Ly = \int_a^b k(x,t)y(t)\, dt. \tag{78}$$

Then

$$\langle Ly, w\rangle = \int_a^b \left(\int_a^b k(x,t)y(t)\mathrm{d}t\right)\overline{w(x)}\mathrm{d}x$$

[10] think: what might you do if the βs were trig functions?

$$= \int_a^b \int_a^b k(x,t)\overline{w(x)}y(t)\mathrm{d}t\mathrm{d}x \qquad \text{double integral}$$
$$= \int_a^b y(t) \int_a^b k(x,t)\overline{w(x)}\mathrm{d}x\mathrm{d}t \qquad \text{switch order}$$
$$= \int_a^b y(t) \int_a^b \overline{\overline{k(x,t)}w(x)}\mathrm{d}x\mathrm{d}t \qquad \text{switch order}$$
$$= \langle y, L^* w\rangle$$

$$\Rightarrow L^* w = \int_a^b \overline{k(x,t)}w(x)\mathrm{d}x$$

Note that we integrate with respect to x! Thus, if $k(x,t)$ is real, the kernel for the adjoint is $k(t,x)$ (as opposed to $k(x,t)$)!
We thus have

$$L^* w = Lw \Leftrightarrow k(x,t) = k(t,x).$$

Real symmetric kernel $\Rightarrow$ **self adjoint**, an instant visual check!

Inhomogeneous equations

Let's consider $Ly = f(x)$ for a 1st kind FIE with a degenerate kernel, i.e.

$$Ly = \sum_{j=1}^{n} \alpha_j(x) \left(\int_a^b \beta_j(t) y(t)\, dt \right) = f(x) \tag{79}$$

Let's begin with the question of existence and uniqueness. Using similar ideas as above (Eq (77)), we can always construct non-trivial solutions to $Ly = 0$ (and to $L^* y = 0$). Thus, we are not going to have uniqueness. What about existence?

There are two ways we might go about this:

Visual route.
Since each $\int \beta_j y\, dt$ will integrate to a constant, the equation is of the form

$$c_1\alpha_1(x) + c_2\alpha_2(x) + \cdots + c_n\alpha_n(x) = f(x)$$

It is clear that this can only be satisfied if $f(x)$ is in the span of the α_js, i.e.

$$\text{the solvability condition is} \quad f \in \text{span}\{\alpha_j\}$$

That seemed too easy. Let's make sure it agrees with all this FAT stuff we know...

FAT route.

1. As we've seen, the adjoint operator has the same kernel, but with the variables switched, so that

$$L^*w = \sum_{j=1}^{n} \beta(t) \left(\int_a^b \alpha(x)y(x)\, dx \right) \tag{80}$$

2. Thus, $L^*w = 0$ has solutions $w_{0,k}$ that are perpendicular to each of the α_j

3. FAT says the solvability condition is that f must be orthogonal to all the $w_{0,k}$

4. Since the $w_{0,k}$ are orthogonal to all of the α_js, for f to be orthogonal to all the $w_{0,k}$ we would need f to be in the span of the α_j; that is we again conclude

$$\text{the solvability condition is} \quad f \in \text{span}\{\alpha_j\}$$

To summarize, we've seen that a 1st kind FIE will never have a unique solution, and existence breaks down into whether or not we can write f as

$$f = \sum_{i=1}^{n} d_i \alpha_i(x)$$

for some constants d_i. If not, no solution is possible.

If so, if $f \in \text{span}\{\alpha_j\}$, then solutions exist, but they are not unique, because we can add arbitrary solutions of the homogeneous problem $Ly = 0$.

<u>How to find solutions?</u>

Writing $f = \sum_{i=1}^{n} d_i \alpha_i(x)$ (where the d_i are known) and noting that the coefficient of $\alpha_j(x)$ in Ly is $\int_a^b \beta_j y\, dt$, we see that

$$\int_a^b \beta_j(t)y(t)\, dt = d_j \tag{81}$$

must hold for all $j = 1, 2, \dots n$

Next, we assume that $y \in \text{span}\{\beta_j\}$. How? Well, any part of y that were not in the span would be orthogonal to the β_j and would thus vanish from (81). Hence, we make the ansatz

$$y(x) = \sum_{i=1}^{n} s_i \beta_i(x) \tag{82}$$

and our task is completed if we can find the s_i. To do this, plug and chug. Stick y from (82) into (81), and we get an $n \times n$ linear system for the s_i:

$$\sum_{i=1}^{n} s_i \int_a^b \beta_j(t)\beta_i(t)\mathrm{d}t = d_j \qquad j = 1, \dots, n$$

Moreover, one can show that the coefficient matrix $(\int_a^b \beta_j \beta_i)_{j,i}$ is invertible if $\{\beta_1, \dots, \beta_n\}$ are linearly independent functions.

Example

$$Ly = \int_0^1 k(x,t)y(t)\mathrm{d}t = 8x - 7x^2$$

with

$$k(x,t) = 4xt - 5x^2t^2 \qquad (\alpha_1 = x,\ \alpha_2 = x^2, \beta_1 = 4t, \beta_2 = -5t^2)$$

Ansatz: $y(x) = s_1 x + s_2 x^2$ (can absorb the constants 4, -5 into s_1, s_2)

$$\text{so} \quad \int_0^1 \beta_1(t)y(t)\ dt = d_1 \Rightarrow\ s_1\frac{4}{3} + s_2 = 8$$

$$\text{and} \quad \int_0^1 \beta_2(t)y(t)\ dt = d_2 \Rightarrow\ s_1\left(\frac{-5}{4}\right) - s_2 = -7$$

$$\Rightarrow s_1 = 12,\ s_2 = -8$$

$$\Rightarrow y(x) = 12x - 8x^2$$

2nd kind FIE

Fredholm Integral Equations of the 2nd kind:

$$Ly = ry(x) + \int_a^b k(x,t)y(t)\,dt = f(x)$$

A 2nd kind FIE only differs from a 1st kind FIE by the extra term $ry(x)$

$$L_1 y = \int_a^b k(x,t)y(t)\mathrm{d}t \qquad 1^{st} \text{ kind}$$

$$L_2 y = L_1 y + ry \qquad 2^{nd} \text{ kind}$$

Thus, if λ_1 is an eigenvalue of L_1 with eigenfunction y, so that $L_1 y = \lambda_1 y$ then

$$L_2 y = L_1 y + ry = (\lambda_1 + r)y$$

which implies that $\lambda_2 = \lambda_1 + r$ is an eigenvalue of L_2. In other words,

The 2nd kind FIE has the same eigenfunctions, with eigenvalues shifted by r

Example

$$k(x,t) = 4xt - 5x^2t^2, \qquad r = \frac{1}{2}$$

$$Ly \equiv \int_a^b k(x,t)y(t)\mathrm{d}t + \frac{1}{2}y$$

We already found the eigenvalues to the equivalent 1st kind FIE to be 1/2 and $-1/6$, plus the zero eigenvalue with infinite multiplicity. Thus the 2nd kind with $r = 1/2$ has eigenvalues 1, $\frac{1}{3}$ (mult. 1 each), $\frac{1}{2}$ (mult. ∞!)

There is no zero eigenvalue, so FAT tells us that $Ly = f$ has a unique solution for all f.

Imhomogeneous 2nd kind FIE with degenerate kernel

How do we solve $Ly = f$ for a 2nd kind FIE for separable $k = \sum \alpha_j \beta_j$? Here, we can get an ansatz immediately by playing the visual game:

$$\sum_j \alpha_j(x) \left(\int_a^b \beta_j(t)y(t)\,dt \right) + ry(x) = f(x) \tag{83}$$

$\Rightarrow y(x)$ has the form

$$y(x) = \frac{1}{r}f(x) - \frac{1}{r}\sum_{j=1}^{n} s_j\alpha_j(x). \tag{84}$$

Notice that the ansatz for is different from the ansatz for a 1st kind FIE.

To obtain a linear system of equations for the s_j, plug (84) into (83), evaluate the integrals and compare the coefficients of the α_j.

$$\Rightarrow \quad \sum_j s_j \int_a^b \beta_k(t)\alpha_j(t)\mathrm{d}t + rs_k = \int_a^b \beta_k(t)f(t)\mathrm{d}t$$

for $k = 1, \ldots, n$. This is a $n \times n$ system for vector $s \equiv (s_1, \ldots, s_n)$, which we can write in matrix form

$$(A + rI)s = b,$$

$$A = (a_{kj}), \quad \text{with } a_{kj} = \int_a^b \beta_k(t)\alpha_j(t)\mathrm{d}t, \qquad j, k = 1, \ldots, n,$$

$$b = (b_1, \ldots, b_n), \quad \text{with } b_k = \int_a^b \beta_k(t)f(t)\mathrm{d}t.$$

<u>Example</u>

$$K(x,t) = 4xt - 5x^2t^2$$

$$\int_0^1 Ky\mathrm{d}t + \frac{1}{2}y = \underbrace{8x - 7x^2}_{f(x)}; \quad r = \frac{1}{2}$$

$$Y = \underbrace{2}_{\frac{1}{r}}\underbrace{(8x - 7x^2)}_{f} - \underbrace{2}_{\frac{1}{r}}\underbrace{(s_1x + s_2x^2)}_{\sum s_j\alpha_j}$$

$$\alpha_1 = x,\ \alpha_2 = -x^2,\ \beta_1 = 4t,\ \beta_2 = -5t^2$$

$$A = \begin{pmatrix} \frac{4}{3} & 1 \\ \frac{-5}{4} & -1 \end{pmatrix} \Rightarrow A + \frac{1}{2}I = \begin{pmatrix} \frac{11}{6} & 1 \\ \frac{-5}{4} & \frac{-1}{2} \end{pmatrix}$$

$$b_1 = \int_0^1 4t(8t - 7t^2)\mathrm{d}t = \frac{11}{3}$$

$$b_2 = \int_0^1 -5t(8t - 7t^2)\mathrm{d}t = -3$$

$$\Rightarrow s_1 = \frac{7}{2} \qquad s_2 = \frac{-11}{4}$$

$$y = 16x - 14x^2 - 7x + \frac{11}{2}x^2 = 9x - \frac{17}{2}x^2$$

And this $y(x)$ is unique!

Summary:

I 1st kind FIE

- The homogeneous equation $Ly = 0$ has a large solution space
- Thus $Ly = f$ has a (very) non-unique solution for very restricted $f(x)$, otherwise non-solvable.

II 2nd kind FIE:

- The eigenvalues are just shifted from the eigenvalues of the equivalent 1st kind FIE.
- If the homogeneous equation $Ly = 0$ has only trivial solutions, then there is a unique solution for all f.
- If $Ly = 0$ has non-trivial solutions, it is still a finite-dimensional solution space, say of dimension N. Then there is no unique solution to $Ly = f$, i.e. depending on f, there is either no solution or an N-dimensional subspace of solutions.
- The N solvability conditions are obtained by forming N inner products of f with a basis of the null-space of the adjoint operator.

Non-degenerate/nonseparable kernels

Many physical systems involve FIE with a non-generate kernel, but that is still real and symmetric. Here, we consider a particular and common class of non-separable kernels, called Hilbert-Schmidt kernels.

A **Hilbert-Schmidt** (HS) kernel $k(x,t)$ has the properties of being real valued, symmetric, and continuous. An integral operator

$$\int_a^b k(x,t)y(t)\,\mathrm{d}t$$

with a HS kernel is called a Hilbert-Schmidt integral operator. HS theory for nondegenerate *self-adjoint* FIE is similar to Sturm-Liouville theory for self-adjoint ODE-BVP. In particular,

Properties of HS-FIE

(1) All eigenvalues are real and non-zero;

(2) Eigenvalues form a discrete infinite set $\{\lambda_j\},\ j = 1, 2, \dots$

$$|\lambda_1| \geq |\lambda_2| \geq \dots$$

$\lim_{j\to\infty} \lambda_j = 0$; each $\lambda_j \neq 0$ has finite multiplicity.

(3) Eigenfunctions for $\lambda_j \neq \lambda_k$ are orthogonal

$$\int_a^b y_j(x) y_k(x) \mathrm{d}x = 0$$

(4) Eigenfunctions are a complete set, i.e. any h can be expanded as

$$h(x) = \sum \frac{\langle h, y_k \rangle}{\langle y_k, y_k \rangle} y_k(x)$$

(5) Solutions to $Ly = f$ can be understood via FAT and where suitable constructed via an eigenfunction expansion.

Note: while the structure is nice, HS theory does not provide any surefire way to find eigenfunctions. This fact is also true for SL theory!

Example:
Let $k(x,t)$ be the function

$$g(x,t) = \begin{cases} (1-t)x & x \leq t \\ (1-x)t & x > t \end{cases}$$

This is a HS-kernel, thus

$$Ly \equiv \int_0^1 k(x,t) y(t) dt.$$

is a HS-operator. Suppose we want to solve the 2nd kind FIE

$$Ly + ry = f \tag{85}$$

where r is a given constant and $f(x)$ a given function. First, consider the eigenfunction problem

$$Ly = \mu y. \tag{86}$$

One approach is to convert to an ODE system. Differentiating across, we obtain

$$y'' + \frac{1}{\mu} y = 0. \tag{87}$$

We can also easily infer the BC's

$$y(0) = y(1) = 0 \tag{88}$$

This system is easily solved, giving eigenvalues/functions

$$y_k = \sin(k\pi x), \quad \mu_k = \frac{1}{k^2\pi^2} \tag{89}$$

We now have all we need to set up solvability conditions for (85).
Alternatively, suppose we had noticed that k is the Green's function of the SL operator

$$\hat{L}y \equiv -y''(x), \qquad y(0) = 0 = y(1).$$

The eigenvalue problem $\hat{L}y = \lambda y$,

$$y'' + \lambda y = 0 \quad \text{on} \quad 0 < x < 1, \qquad y(0) = 0 = y(1)$$

is equivalent, and the eigenvalues/eigenfunctions are

$$\lambda_k = k^2\pi^2, \quad y_k = \sin k\pi x \quad k = 1, 2, \ldots$$

But we can also write solutions of $\hat{L}y = \lambda y$ in terms of the Green's function (with λy playing the role of $f(x)$):

$$y = -\lambda \int_0^1 k(x,t)y(t)\mathrm{d}t$$

and of course we get the same eigenvalues and eigenfunctions.
Thus, considering the eigenvalue problem for the HS-Operator L:

$$\int_0^1 k(x,t)y(t)dt = \mu y,$$

we can infer the eigenvalues/functions:

$$\mu_k = -\frac{1}{\lambda_k} = \frac{1}{k^2\pi^2}, \quad y_k = \sin k\pi x \quad k = 1, 2, \ldots.$$