Fredholm Alternative

We have now seen two different ways to solve linear BVPs. So far, we have been happily going along, assuming that the solutions we construct are *the, one and only*, solutions. But are we sure? Now it is time to address the important question of existence and uniqueness.

Existence: Does a solution actually exist? Will the method we employ "work," or will it for instance lead to contradictions?

Uniqueness: We've found a solution, it exists, great. Is it the only one?

From a mathematical point of view, these are incredibly important questions. They can be even more important from a physics/applied mathematics standpoint. If you're modelling a physical situation with a particular

differential equation, you'd like to think that you can find a single answer that has physical meaning and that would agree with experiment.
To think about: What would non-existence mean in a physical context? What about non-uniqueness?

These are important questions, and whole branches of mathematics have been developed to think about them. Here, we'll just look at one very useful theorem in this arena: the Fredholm Alternative Theorem (FAT).

A closer look at SL solution

Let's look more closely at the solution we obtained in Section 2.8.1 for a SL-BVP. The last step in obtaining the coefficients c_k of

$$y = \sum_k c_k y_k$$

was

$$\lambda_k c_k \langle y_k, r y_k \rangle = \langle f, y_k \rangle. \tag{71}$$

But what happens if there is an eigenvalue equal to zero, say $\lambda_0 = 0$? We potentially have a problem, as this equation would read

$$0 \times c_0 = \langle f, y_0 \rangle.$$

If so, there are two possibilities:

1. $\langle f, y_0 \rangle \neq 0$. In this case, we have a contradiction, and we are forced to conclude that no solution exists.

2. $\langle f, y_0 \rangle$ does actually equal zero. In this case, no contradiction, but we don't get any way to solve for c_0 either. Hence

$$y = c_0 y_0(x) + \sum_{k=1}^{\infty} c_k y_k(x)$$

where c_k are calculated values for $k = 1, 2, \ldots$, but c_0 is arbitrary, leading to an *infinite set of solutions*.

So, if there is a zero eigenvalue (meaning $\lambda_0 = 0$ yields a non-trivial solution), it seems that either we get non-existence, or we get existence but non-uniqueness.

And if there isn't a zero eigenvalue? If there is no eigenfunction corresponding to $\lambda_0 = 0$, then (71) would never yield any problems, we could always divide by λ_k to get well defined c_k, and the expansion would have no issues. Thus, we would have both existence and uniqueness.

Zero eigenvalue

The question of the zero eigenvalue is a special case, as we are really asking whether the *homogeneous* system

$$Ly = 0$$

has a non–trivial solution $y_0(x)$.

FAT

What we've just seen above for a SL-BVP in fact holds much more widely. For general (not necessarily self-adjoint) ODE-BVPs, the statement of FAT reads:

Exactly one of the two alternatives holds:

I. **EITHER** the homogeneous adjoint problem

$$L^* w_0 = 0, \quad BC_1^* = 0, \ BC_2^* = 0$$

has a non-trivial solution

II. **OR** the inhomogeneous problem

$$Ly = f, \quad BC_1 = \alpha_1, \ BC_2 = \alpha_2$$

has a unique solution for any f, α_1, α_2

Notes:

1. Note the exclusive "or". Exactly one of the alternatives is true.
2. You'll see I've put in inhomogeneous boundary conditions. If we are in Case II, with no homogeneous solution, then we can construct a solution with inhomogeneous BC either by decomposing or by putting BC directly into the eigenfunction construction.

In Case I, there are two subcases. Suppose we have inhomogeneous BCs, and we solve the problem by putting the BC's directly into the eigenfunction expansion. Since there is an eigenvalue $\lambda_0 = 0$, we will have an equation that looks like:

$$0 \times c_0 = \langle f, w_0 \rangle + \text{'stuff'}_0$$

where 'stuff'$_0$ has come from the inhomogeneous boundary conditions. The two possibilities are:

1. If the RHS is non-zero, then we have a contradiction, and thus no solution exists.

2. If the RHS=0, there is no contradiction, but we have no information on c_0 either, thus the solution is valid for any constant c_0, and we have existence but non-uniqueness.

Homogeneous vs inhomogeneous BC

With homogeneous boundary conditions, FAT has a very 'nice' form, easily stated in words. It says that to have a unique solution, the adjoint homogeneous problem must have only the trivial solution; otherwise, if there is a non-trivial solution w_0, then the *solvability condition* to have a solution (but non-unique!) is that the forcing function $f(x)$ must be orthogonal to the homogeneous solution w_0.

With inhomogeneous BC, the criteria for a unique solution is the same, but when the non-trivial w_0 exists, the solvability condition is not so cute, due to the 'stuff'$_0$ arising from the BCs. Note that if we had tried to decompose the solution, i.e. separate the boundary conditions into a problem $Ly = 0,\ BC \neq 0$, then in looking at $Ly = f,\ BC = 0$, we would arrive at the wrong solvability condition, since there would be no mention of boundary conditions! In other words, decomposing the solution could lead to incorrect conclusions! Which leads to the following...

Health warning: If you have a problem with *inhomogeneous boundary conditions* AND there is a *zero eigenvalue*, do not try to decompose the solution. Incorporate the boundary conditions directly into the eigenfunction expansion!

Examples

Ex. 1
Solve $y'' + y = f$, $y(0) = 0$, $y(\pi) = 0$. This is self-adjoint and has the zero eigensolution $y_0 = \sin x$. Then

$$\begin{aligned}\langle y'', \sin x\rangle &= \int_0^{\pi} y'' \sin x \mathrm{d}x \\ &= -\int_0^{\pi} y \sin x \mathrm{d}x \quad \text{(by parts, twice)} \\ &= -\langle y, \sin x\rangle\end{aligned}$$

There is a solution only if

$$\langle f, \sin x\rangle = 0,$$

in which case $y + c \sin x$ is a solution for all c.

Ex. 2a

Solve $y'' = f(x)$ with $0 < x < 1$, $y(0) = 0$ and $y'(1) = 7$. The zero eigenvalue adjoint problem is $L^* w_0 = w_0''(x) = 0$ (this problem is also self-adjoint), with $w_0(0) = w_0'(1) = 0$. This only has the trivial solution $w_0 \equiv 0$, so the full problem has a unique solution for any $f(x)$.

Ex. 2b

Same problem, but change the BC to $y'(0) = 0$ and $y'(1) = \beta$, and let $f(x) = 3$. This time we get $w_0(x) = 1$. Thus:

$$\begin{aligned}\langle y'', w_0\rangle &= \langle f, w_0\rangle \\ \Rightarrow \int_0^1 y'' \mathrm{d}x &= 3 \\ \Rightarrow y'|_0^1 = 3 &\end{aligned}$$

The BC's give that $y'(1) - y'(0) = \beta$, and thus if $\beta \neq 3$, we have a contradiction and no solution exists, while if $\beta = 3$, we have a non-unique solution.

FAT - Linear algebra version

The Fredholm alternative can also be expressed as a theorem of linear algebra. It addresses the question: when does $Ax = b$ have a unique solution?, where

$$\left.\begin{array}{l} A \in \mathbb{R}^{n,n} : n \times n \text{ matrix} \\ b \in \mathbb{R}^n : n \text{ dim, column vector} \end{array}\right\} \text{given}$$

$$x \in \mathbb{R}^n : n \text{ dim column vector (unknown)}$$

FAT says:
Exactly one of the two alternatives holds:

I. Either: $A^T y = 0$ has a non-trivial solution $y \neq 0$.

II. Or: $Ax = b$ has a unique solution (has a solution, and it is unique).

If $A^T y = 0$ has solutions $y \neq 0$, then $Ax = b$ has either no or multiple solutions.
Distinguish cases by solvability condition.
Let $y_1, y_2, \ldots, y_N$ be a basis of $A^T y = 0$.

(a) If $y_k^T b = 0$ for all k, then $Ax = b$ is solvable, and the solution space has dimension N.

(b) If $y_k^T b \neq 0$ for one or more k, then $Ax = b$ has no solutions.

Example

(1)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Look at $A^T y = 0$ (GE=Gaussian Elimination)

$$\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 4 & 0 \end{array}\right) \xrightarrow{\text{GE}} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \end{array}\right) \Rightarrow \begin{array}{ll} y_1 + 3y_3 & = 0 \\ -2y_2 & = 0 \end{array}$$

$\Rightarrow y = 0$ i.e. $A^T y = 0$ only has a trivial solution. Therefore, FAT I is false, thus FAT II is true i.e. $Ax = b$ has a unique solution (for any b).

(2)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A^T y = 0 : \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right) \xrightarrow{\text{GE}} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow y_1 + 3y_2 = 0$$

Thus, a non-trivial solution for $A^T y = 0$ is: $y_0 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

Therefore I is true, thus II of FAT is false:

$Ax = b$ has either none or multiple solutions. Which of these two cases applies? That depends on b. Since y_0 forms a basis for the one dimensional null space $A^T y = 0$, the solvability condition is $y_0^T b = 0$.

Alternatively, we could see this by GE for $Ax = b$.

$$\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 6 & b_2 \end{array}\right) \xrightarrow{\text{GE}} \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 3b_1 \end{array}\right) \Rightarrow \begin{array}{ll} x_1 + 2x_2 & = b_1 \\ 0 & = b_2 - 3b_1 \end{array}$$

So, no solution if $b_2 - 3b_1 \neq 0$, and a one-dimensional space of solutions if $b_2 - 3b_1 = 0$.
Note:

$$b_2 - 3b_1 = (-3, 1)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = y^T b$$

FAT and Green's functions

The Green's function approach gave the solution to $Ly = f$ as

$$y = \int_a^b g(x, \xi) f(\xi)\, d\xi.$$

So, if the GF approach works, i.e. if we can find g, then we have both existence and uniqueness. *How does this fit in with FAT?* Clearly, if there is a zero eigenvalue, something must go wrong in the construction of the GF. Consider the formulation for the GF:

$$Lg(x, \xi) = \delta(x - \xi),$$

and let's apply FAT, thinking of ξ as a dummy variable. If there is a zero eigenvalue, then there exists non-trivial $y_0(x)$ for which $Ly_0 = 0$. Then we are in case I and $Lg = \delta$ does not have a unique solution, and the solvability condition for any solution is

$$\langle y_0(x), \delta(x - \xi) \rangle = 0$$

which clearly does not hold since

$$\langle y_0(x), \delta(x - \xi) \rangle = y_0(\xi)$$

and $y_0 \not\equiv 0$. Thus, we can't construct the GF.

Modified Green's function

The above analysis suggests that something should go wrong when attempting to construct a GF in the case of a zero eigenvalue. However, we know that a non-unique solution still exists if the solvability condition is satisfied. In that case, we can construct a *modified Green's function*. The modified GF $g_m(x, \xi)$ satisfies

$$Lg_m(x, \xi) = \delta(x - \xi) - \frac{y_0(x)w_0(\xi)}{n_0} \tag{72}$$

where y_0 and w_0 are the eigenfunctions of L and L^* for $\lambda_0 = 0$, and $n_0 = \langle y_0, w_0 \rangle$. One way to see the motivation for this is that the RHS of (72) is now orthogonal to w_0, so the solvability condition is satisfied and one can construct g_m in the normal way, just accounting for the extra term on the right. That is true regardless of the RHS of the system $Ly = f$. However, if we take the solution

$$y(x) = \int_a^b g_m(x, \xi) f(\xi)\, d\xi$$

and apply L to both sides, we find $Ly = f$ only if w_0 is orthogonal to f, consistent with FAT.