Distributions

We've seen in Section 3 how the Green's function can be a valuable tool in solving BVPs. However, constructing the GF required defining the "delta function", which is not really a function at all, at best a limit of functions, and also saw that the GF suffers a discontinuity in the $n-1$st derivative. We now take a short detour to consider these issues in more detail, by introducing the theory of distributions.

Perhaps the most important feature of the δ-"function": when integrated against a continuous function, it sifts out the value at $x=0$:

$$\int_{-\infty}^{\infty} \delta(x) f(x) \mathrm{d}x = f(0).$$

It is the operation of δ on another function that defines the property. This is the key idea in the theory of distributions, in which a generalized function is only thought of in relation to how it affects other functions when "integrated" against them.

We define the delta distribution δ such that when it operates on a test function ϕ, it "sifts out" the value $\phi(0) \in \mathbb{R}$.
We write this as

$$\langle \delta, \phi \rangle \equiv \phi(0),$$

where δ is the δ-distribution and ϕ is the test function. $\langle \delta, \phi \rangle$ reads as "δ applied to ϕ".
We will generalise this idea momentarily. First, we need some tools and terminology.

Test functions $\phi : \mathbb{R} \to \mathbb{R}$
$\phi \in C_0^{\infty}(\mathbb{R})$, which is short for:

- $\phi \in C^{\infty}(\mathbb{R})$ differentiable any number of times
- ϕ has "compact support", i.e. supp $\phi \subseteq [-X, X]$ for some $X > 0$, i.e. $\phi(x) = 0 \quad \forall x \notin [-X, X]$.

So a test function is infinitely smooth, has no kinks or corners, and vanishes outside a finite region.

Example (see figure):

Let $C > 0,\ \epsilon > 0$

$$\phi_{C;\epsilon} = \begin{cases} \exp\left(\frac{-C}{\epsilon^2-(x-a)^2}\right) & \text{for } a-\epsilon < x < a+\epsilon \\ 0 & \text{otherwise} \end{cases} \tag{64}$$

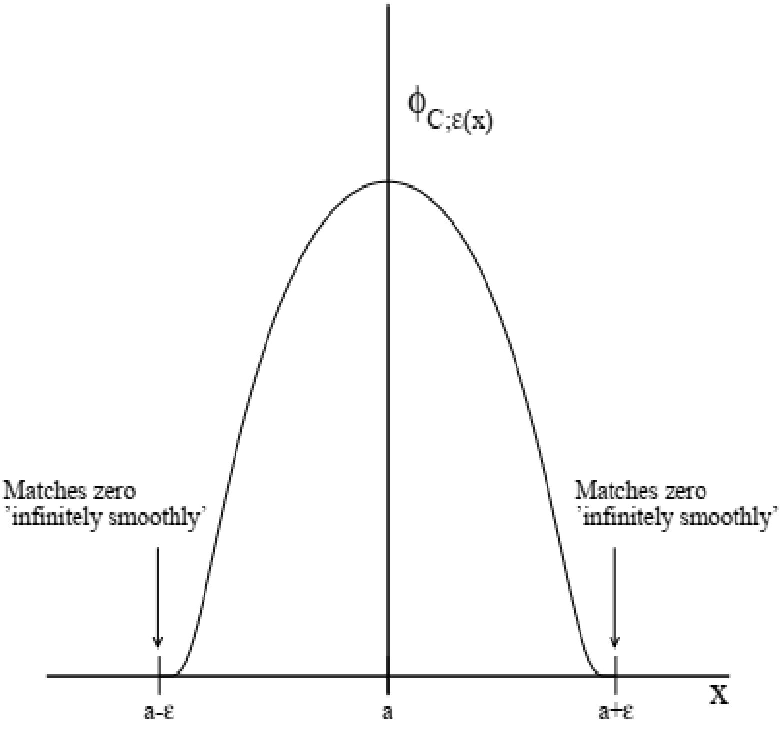

Figure 4: Sample test function, corresponding to (64)

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One can show (for all integer $n \geq 0$):

$$\lim_{x\uparrow a+\epsilon} \frac{d}{dx^n}\phi_{C;\epsilon}(x) = 0$$

$$\lim_{x\downarrow a-\epsilon} \frac{d}{dx^n}\phi_{C;\epsilon}(x) = 0$$

Weak derivative

Having defined test functions, we can generalise the notion of a derivative. Start with the classical definition: let $u(x)$ be a continuously differentiable function with derivative $f(x)$, so $u'(x) = f(x)$. Now, multiply each side of the equation by a test function ϕ and integrate over $\mathbb{R}$:

$$\int_{\mathbb{R}} u'\phi \, dx = \int_{\mathbb{R}} f\phi \, dx. \tag{65}$$

Integrating the LHS by parts and using the compact support of ϕ, we obtain

$$-\int_{\mathbb{R}} u\phi' \, dx = \int_{\mathbb{R}} f\phi \, dx. \tag{66}$$

The idea of the weak derivative is to think of (66) as the *definition* of a derivative. That is, we say f is the *weak derivative* of u if (66) holds for all test functions $\phi \in C_0^\infty(\mathbb{R})$[7]. The value is that this definition does not require u to be differentiable, just integrable. Of course, if u is continuously differentiable, the weak derivative and the ordinary one will agree, but a function that is not continuously differentiable can still have a weak derivative, where essentially the integration smooths out discontinuities.

Distribution definition

This leads us to the notion of a distribution, or a generalised function. A distribution is not defined at points, but rather it is a global object defined in terms of its action on test functions. To be more precise:
Definition: A distribution u is a functional mapping test functions $\phi \in C_0^\infty(\mathbb{R})$ to real numbers,

$$u : \phi \in C_0^\infty(\mathbb{R}) \mapsto \langle u, \phi \rangle \in \mathbb{R} \qquad (\langle u, \phi \rangle \text{ instead of } u(\phi)) \tag{67}$$

where the mapping is linear and continuous. While we have motivated the action $\langle u, \phi \rangle$ as meaning integration, this is not a requirement.

Linearity is straightforward, and means

$$\langle u, \alpha\phi + \beta\psi \rangle = \alpha\langle u, \phi \rangle + \beta\langle u, \psi \rangle \qquad \forall \alpha, \beta \in \mathbb{R} \quad \forall \phi, \psi \in C_0^\infty(\mathbb{R}) \tag{68}$$

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[7] We can also confine to smaller intervals, for instance $\phi \in C_0^\infty(a, b)$ means the test functions have compact support in a bounded subset of (a, b).

Continuity is slightly more technical, it means that if ϕ_n is a sequence of test functions that converges to zero,

$$\phi_n(x) \to 0 \quad \text{as} \quad n \to \infty$$

then

$$\langle u, \phi_n \rangle \to 0 \tag{69}$$

as a sequence of real numbers.
To show continuity, what we really need is to be able to switch the order of "the action of the distribution" (integration) and the limit, that is (69) will hold if

$$\lim_{n\to\infty} \langle u, \phi_n \rangle \;=\; \langle u, \lim_{n\to\infty} \phi_n \rangle.$$

It turns out that we can do this if the following holds:

(*) $\forall X > 0$ there exists $C > 0$, and integer $N \geq 0$, such that

$$|\langle u, \phi \rangle| \leq C \sum_{m \leq N} \max_{-\infty \leq x \leq \infty} \left| \frac{\mathrm{d}^m \phi}{\mathrm{d}x^m} \right|$$

$\forall \phi$ with support in $[-X, X]$.

For our purposes we will want to show (*) to show continuity, and in fact you can take this as the definition of continuity.

Examples

Delta distribution

$$\langle \delta, \phi \rangle = \phi(0)$$

linearity: ✓

continuity, check (*): $|\langle \delta, \phi \rangle| = |\phi(0)| \leq \max_{-X<x<X} |\phi(x)| \quad \forall \phi$ with support of ϕ in $[-X, X]$.
i.e. condition (*) is satisfied with $C = 1$, $N = 0$.
Generalisation
Let $a \in \mathbb{R}$, $n \geq 0$. Define $\langle D_n, \phi \rangle = \phi^{(n)}(a)$ (nth derivative).
This is a distribution (to be proved in a problem sheet).

Functions as distributions. For any locally integrable function $f(x)$, a natural distribution is defined by

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\mathrm{d}x$$

Check:

Well-defined, $\langle f, \phi \rangle \in \mathbb{R} \, \forall \phi \in C_0^\infty(\mathbb{R})$ and linear.

Continuity? (*): Let $X > 0$ be given. Claim (*) holds for

$$C = C(X) = \int_{-X}^{X} |f(x)|\mathrm{d}x \text{ and } N = 0 :$$

$$|\langle f, \phi \rangle| = |\int_{-\infty}^{\infty} f(x)\phi(x)\mathrm{d}x| = |\int_{-X}^{X} f(x)\phi(x)\mathrm{d}x|$$

which by the estimation lemma

$$\leq \int_{-X}^{X} |f(x)|\mathrm{d}x \max_{-X<x<X}(|\phi(x)|) = C \max_{-\infty<x<\infty}(|\phi(x)|)$$

Remark: Different *continuous* functions induce different distributions.

Heaviside function $H(x)$

$$\langle H, \phi \rangle = \int_{\infty}^{\infty} H(x)\phi(x)\mathrm{d}x = \int_{0}^{\infty} \phi(x)\mathrm{d}x$$

Can check linearity, continuity as an exercise.

Remark: Different functions can lead to the same distribution.
Distributions induced by *integrable* functions are called *regular* distributions; singular distributions if not. The δ-distribution is an example of a singular distribution.

Operations on distributions

Now we consider some operations that can be performed on distributions. Let u_1, u_2, u be *distributions*, and f_1, f_2, f be integrable *functions* (or the regular distributions induced by them). The notion of integration is not required for distributions, but the rules for distributions are consistent with those for locally integrable functions.

Linear combinations of distributions. Let $\alpha_1, \alpha_2 \in \mathbb{R}$.

$$\begin{aligned}\langle \alpha_1 f_1 + \alpha_2 f_2, \phi\rangle &= \int_{-\infty}^{\infty} (\alpha_1 f_1(x) + \alpha f_2(x))\phi(x)\mathrm{d}x \\ &= \alpha_1 \int_{-\infty}^{\infty} f_1(x)\phi(x)\mathrm{d}x + \alpha_2 \int_{-\infty}^{\infty} f_2(x)\phi(x)\mathrm{d}x \\ &= \alpha_1\langle f_1, \phi\rangle + \alpha_2\langle f_2, \phi\rangle\end{aligned}$$

Thus, define $\alpha_1 u_1 + \alpha_2 u_2$ for general distributions u_1, u_2 via

$$\langle \alpha_1 u_1 + \alpha_2 u_2, \phi\rangle \equiv \alpha_1\langle u_1, \phi\rangle + \alpha_2\langle u_2, \phi\rangle \quad \forall \phi \in C_0^\infty(\mathbb{R})$$

If u_1, u_2 are distributions, is $\alpha_1 u_1 + \alpha_2 u_2$ a distribution? Need to check linearity and continuity, but we'll skip this here.

Differentiation of distributions. Differentiation follows the weak derivative formulated earlier. That is, for a general distribution u, define

$$\langle u', \phi\rangle \equiv -\langle u, \phi'\rangle \qquad \forall \phi \in C_0^\infty(\mathbb{R})$$

If u is distribution, can we be sure that $u' : \phi \mapsto -\langle u, \phi'\rangle$ is also a distribution? (It is! – try it as an exercise.)

Example. Let H be the Heaviside function, or the distribution it induces, i.e.

$$\langle \underbrace{H}_{H\text{-distribution}}, \phi\rangle \equiv \int_{-\infty}^{\infty} \underbrace{H(x)}_{H-\text{function}} \phi(x)\mathrm{d}x = \int_0^{\infty} \phi(x)\mathrm{d}x$$

Show that $H' = \delta$.

$$\begin{aligned}\langle H', \phi\rangle &= \langle -H, \phi'\rangle && \text{Def. of derivative of a distribution} \\ &= \textstyle\int_{-\infty}^{\infty} \phi'(x)\mathrm{d}x && \text{see earlier example} \\ &= -\phi|_{x=0}^{x=\infty} \\ &= \phi(0) && \phi \text{ has compact support} \\ &= \langle \delta, \phi\rangle && \text{Def. of } \delta\text{-distribution}\end{aligned}$$

Translation: similar considerations as before, upshot ($a \in \mathbb{R}, u$ distr):

$$\langle u(x-a), \phi(x)\rangle \overset{\text{chg of var}}{=} \langle u(y), \phi(y+a)\rangle = \langle u(x), \phi(x+a)\rangle$$

Example: $\langle\delta(x-a),\phi(x)\rangle = \langle\delta(x),\phi(x+a)\rangle = \phi(a)$

Multiplication: let $a(x)$ be an infinitely differentiable function. We define

$$\langle au,\phi\rangle = \langle u, a\phi\rangle.$$

Convergence of a sequence of distributions $u, u_1, u_2, \dots$ distributions. Convergence $u_j \to u$ as $j\to\infty$ means:

$$\lim_{j\to\infty}\langle u_j,\phi\rangle = \langle u,\phi\rangle \qquad \forall\phi\in C_0^\infty(\mathbb{R})$$

Similarly: if $u(\alpha)$ is a family of distributions with a continuous parameter α, then
convergence $u(\alpha)\to u(\alpha_0)$ for $\alpha\to\alpha_0$ means:

$$\lim_{\alpha\to\alpha_0}\langle u(\alpha),\phi\rangle = \langle u(\alpha_0),\phi\rangle \qquad \forall\phi\in C_0^\infty(\mathbb{R})$$

Distributed solutions

Consider the equation

$$Lu \equiv a_2u'' + a_1u' + a_0u = f.$$

We have always thought about the *classical solution*, that is a twice continuously differentiable function $u(x)$ that satisfies the differential equation identically, i.e. we can take derivatives of u, substitute in, and the equation checks at every point. With distribution theory and the notion of a generalised function, we now can define a *distributed solution*. That is, if u and f are distributions, then Lu is a distribution, defined by the action

$$\begin{aligned}\langle Lu,\phi\rangle &= \langle a_2u'',\phi\rangle + \langle a_1u',\phi\rangle + \langle a_0u,\phi\rangle \\ &= \langle u,(a_2\phi)''\rangle - \langle u,(a_1\phi)'\rangle + \langle u,a_0\phi\rangle \overset{\text{define}}{=} \langle u, L^*\phi\rangle.\end{aligned} \tag{70}$$

Here L^* is the formal adjoint operator. We say that u is a distributed solution to $Lu = f$ if

$$\langle u, L^*\phi\rangle = \langle f,\phi\rangle$$

holds for all test functions ϕ. We highlight again that a function need not be differentiable in the ordinary sense to satisfy this definition; hence, distributions provide a way to have well-defined solutions that may have issues in the classical sense.

In particular, this construction of a distributed solution gives us a new way to interpret the Green's function. Since δ is really a distribution or a generalised function, the equation $Lg = \delta(x-\xi)$ should be interpreted in the distributional sense,

$$\langle Lg, \phi \rangle = \langle \delta(x-\xi), \phi \rangle$$

or

$$\langle g(x,\xi), L^{*}\phi \rangle = \phi(\xi).$$

Moreover, since the Green's function that we construct is not twice continuously differentiable, it is really a distributed solution. Alternatively, if we interpret $Lg = \delta(x-\xi)$ as meaning that $Lg = 0$ everywhere that $x \neq \xi$, then using the properties of δ we can work purely in the "classical" sense. In fact, the final solution of $Ly = f$, obtained by integration with g, is continuous and a classical solution.

Final thoughts: If you are interested in distribution theory, it is at the core of functional analysis. Moreover, the idea of weak formulations has great use in finite element methods. For us, distribution theory is somewhat of a detour for this course. One could proceed to write things in a distributional sense anytime we encounter a 'delta function', but we can as well recognise delta as the limit of continuous functions and satisfying certain properties, thus in effect translating to a classical system. Unless we are specifically interested in a distributional aspect, the latter will be our approach.