3 Green's function

In this section we will devise an alternative approach to viewing and solving linear BVP, using the so-called Green's function.

3.1 Form of the eigenfunction expansion solution

Consider the form of the final solution obtained through the eigenfunction expansion approach. Taking (20) one step further, we have

$$y(x) = \sum_{k=1}^{\infty} \frac{\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle} y_k(x)$$

(Of course, this requires all $\lambda_k \neq 0$ – we'll treat the case of a zero eigenvalue in Section 5). Let $n_k = \langle y_k, w_k \rangle$ (normalisation), then:

$$\begin{aligned} y(x) &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} \left(\int_a^b f(t) w_k(t) \mathrm{d}t \right) y_k(x) \\ &= \int_a^b \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} w_k(t) y_k(x) \right) f(t) \mathrm{d}t \\ &= \int_a^b g(x,t) f(t) \mathrm{d}t \end{aligned}$$

where

$$g(x,t) = \sum_{k=1}^{\infty} \frac{w_k(t) y_k(x)}{\lambda_k n_k}. \qquad (29)$$

Thus, we have constructed a solution to $Ly = f$ in the form

$$y(x) = \int_a^b g(x,t) f(t)\, dt. \qquad (30)$$

The function $g(x,t)$ is called the Green's function (GF), and the form (29) is an eigenfunction expansion of $g(x,t)$.
Of course, if we knew the Green's function, we would have the solution without any need for the expansion, i.e. no need for the eigenfunctions. The goal in this section is to understand the properties of the GF and how to construct it.

Side note: Observe that if $L = L^*$, then $w_k = y_k$ and:

$$g(x,t) = \sum \frac{1}{\lambda_k n_k} y_k(t) y_k(x)$$

In this case $g(x,t) = g(t,x)$, and we have the important connection between a self-adjoint operator and a symmetric Green's function.

3.2 Inverse of differential operator

A nice way to think of the Green's function is in terms of inverting the differential operator. Think about the familiar equation $\mathbf{A}\vec{x} = \vec{b}$ from linear algebra, to be solved for the unknown vector $\vec{x}$. The solution is given by

$$\vec{x} = \mathbf{A}^{-1}\vec{b},$$

i.e. we find the solution by multiplying the inverse of the linear operator (matrix) by the inhomogeneous term. Once you know the inverse operator, you can solve the problem for any given vector $\vec{b}$. In the context of BVP's, L is a differential operator, so it stands to reason that the inverse operator involve integration, hence the form (30). Constructing the Green's function is analogous to finding the inverse of the matrix, once we have g we can write down the solution (30) for any forcing function $f(x)$.

3.2.1 An example

There are numerous ways to construct a Green's function. We've already seen one: the eigenfunction expansion. Another way that you've probably seen before is via variation of parameters[2]. This approach gives the Green's in a piecewise form.

[2]You will not be tested on this method, but if you would like to review the approach you might check out the textbook *Elementary Differential Equations and Boundary Value Problems*, by Boyce and DiPrima.

Let's take a simple example and look at the behaviour. Consider the BVP:

$$\begin{aligned} &Ly \equiv -y'' = f(x), 0 < x < 1 \\ &y(0) = y(1) = 0 \end{aligned} \tag{31}$$

The GF via variation of parameters is given by

$$g(x,\xi) = \begin{cases} (1-\xi)x & 0 < x < \xi \\ (1-x)\xi & \xi < x < 1. \end{cases} \tag{32}$$

The following properties are easily checked:

- The GF satisfies $Lg = 0$ if $x \neq \xi$
- $g(x,\xi)$ satisfies the boundary conditions
- g is continuous on the whole interval $[0,1]$
- g is differentiable everywhere *except at* $x = \xi$, where it suffers a jump in the derivative.

These properties are in fact always true of the GF of a second order linear operator.[3] To make sense of this, and to build some physical intuition, we shall need the notion of the delta function.

3.3 Green's function via delta function

To fix the context, consider stationary heat conduction in a rod:

$$-y''(x) = f(x) \qquad 0 < x < 1 \tag{33}$$

$$y(0) = 0, \quad y(1) = 0. \tag{34}$$

where $y(x)$ is the temperature field and $f(x)$ is a given heat source density.

3.3.1 Delta function

The function $f(x)$ describes any heat added or removed from the system by the outside world. As a simple scenario, consider a *point heat source*, say located at the middle of the rod. Physically, this would correspond to

[3]Note, however, that the function $y(x)$ satisfying $Ly = f$ *is* continuously differentiable assuming continuously differentiable f, meaning that the integration with $f(x)$ smooths out the discontinuity in g.

applying heat at a single point only. How would we describe such a situation mathematically? What should we use for the function $f(x)$?
The notion of a point source is described by the "delta function" δ, characterised by properties

$$\delta(x) = 0 \quad \forall x \neq 0, \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{35}$$

The first property captures the notion of a point function. The second property constrains the area under the curve (which you might think of as infinitely thin and infinitely high). This is an idealized point source at $x = 0$, a point source at $x = a$ would be given by $\delta(x - a)$.

The problem is that no classical function satisfies (35) (think: any function that is non-zero only at a point is either not integrable or integrates to zero).

3.3.2 Approximating the delta function

One way around this is to replace δ by an approximating sequence of increasingly narrower functions with normalized area, i.e. $f_n(x)$ where

$$\int_{-\infty}^{\infty} f_n(x)\mathrm{d}x = 1 \quad \forall n, \qquad \lim_{n\to\infty} f_n(x) = 0 \quad \forall x \neq 0.$$

Example: "hat" functions

$$f_n(x) = \begin{cases} 0 & \text{for } |x| > 1/n \\ n/2 & \text{for } |x| \leq 1/n \end{cases} \tag{36}$$

You can verify the $f_n(x)$ approach $\delta(x)$ as $n \to \infty$.

3.3.3 Properties of delta function

We have defined δ by (35). We can use the approximating functions to obtain further properties.

Sifting property. What happens when δ is integrated against another function?
Let $f(x)$ be a continuous function, and $F(x) = \int^x f(s)ds$ its antiderivative. Now consider approximating sequences:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\mathrm{d}x = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x-a) f(x)\mathrm{d}x,$$

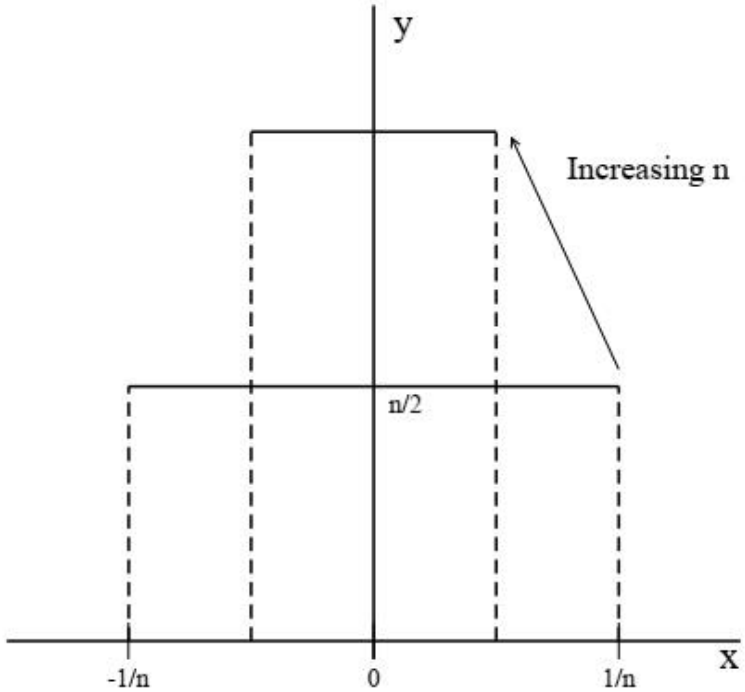

Figure 3: Hat functions, see eqn. (36)

and if f_n are the hat functions (36),

$$= \lim_{n\to\infty} \int_{a-1/n}^{a+1/n} \frac{n}{2} f(x)\mathrm{d}x = \lim_{n\to\infty} \frac{F(a+(1/n)) - F(a-(1/n))}{2/n}$$
$$= \lim_{s\to 0} \frac{F(a+s) - F(a-s)}{2s} = F'(a) = f(a).$$

Thus, we have

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\mathrm{d}x = f(a) \quad \text{if } f \text{ is continuous at } a. \tag{37}$$

In particular,

$$\int_{-\infty}^{\infty} \delta(x) f(x)\mathrm{d}x = f(0) \quad \text{if } f \text{ is continuous at } x = 0. \tag{38}$$

Thus, the delta function can be seen to sift out the value of a function at a particular point.

Antiderivative of $\delta(x)$. The antiderivative of the delta function is the so-called Heaviside function,

$$\int_{-\infty}^{x} \delta(s)\mathrm{d}s = H(x) \equiv \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \tag{39}$$

Note that (39) follows by integrating the sequence of approximating functions and showing that the limit is the Heaviside function. That is, if $H_n(x) = \int_{-\infty}^{x} f_n(s)ds$, then $\lim_{n\to\infty} H_n(x) = H(x)$. (We leave this detail as an exercise!)

3.3.4 Point heat source

Let's return to the heat conduction BVP with a point heat source of unit strength at the centre of the rod:

$$-y''(x) = \delta(x - 1/2), \quad 0 < x < 1 \tag{40}$$

$$y(0) = y(1) = 0. \tag{41}$$

Since $\delta(x - 1/2) = 0 \ \forall x \neq 1/2$, this implies

$$-y''(x) = 0, \quad 0 < x < 1/2, \ 1/2 < x < 1. \tag{42}$$

We can easily solve (42) in each of the two separate domains $[0, 1/2)$ and $(1/2, 1]$ and then apply the BC (41). But be careful: there are two constants of integration for each domain, meaning *four* unknown constants total, and only *two* boundary conditions.
As you might expect (since $\delta(x - 1/2)$ has vanished from (42)), the extra two conditions come in at $x = 1/2$. To derive the extra conditions, imagine integrating equation (40) across $x = 1/2$:

$$\int_{1/2-}^{1/2+} -y''(x)\, dx = \int_{1/2-}^{1/2+} \delta(x - 1/2)\, dx, \tag{43}$$

where $1/2-$ $(1/2+)$ signifies just to the left (right) of $1/2$. Using property (35) of the delta function, we have

$$-\, y'\big]_{1/2-}^{1/2+} = 1 \quad \Rightarrow \quad y'(1/2+) - y'(1/2-) = -1. \tag{44}$$

That is, the presence of the delta function defines a *jump condition* on y'. [4]
The other extra condition needed comes as a requirement that $y(x)$ is continuous across the point source, that is

$$y\big]_{1/2-}^{1/2+} = 0. \tag{45}$$

[4] Here, $y(\xi-) = \lim_{x\uparrow\xi} y(x)$, and $y(\xi+) = \lim_{x\downarrow\xi} y(x)$

More on this condition below. Solving Equations (42), (41) along with extra conditions (44) and (45), we obtain the solution

$$u(x) = \begin{cases} \frac{x}{2} & 0 < x < 1/2 \\ -\frac{x}{2} + \frac{1}{2} & 1/2 < x < 1. \end{cases} \tag{46}$$

3.3.5 Green's function construction

To motivate the construction of the Green's function, consider the heat conduction problem with an arbitrary heat source:

$$-y''(x) = f(x), \quad 0 < x < 1 \tag{47}$$
$$y(0) = y(1) = 0. \tag{48}$$

Imagine now describing f by a distribution of point heat sources with varying strength; that is at point $x = \xi$ we imagine placing the point source $f(\xi)\delta(x - \xi)$.

The idea of the Green's function is to introduce such an extra parameter ξ, and consider the system

$$-g''(x, \xi) = \delta(x - \xi), \quad 0 < x < 1 \tag{49}$$
$$g(0, \xi) = g(1, \xi) = 0. \tag{50}$$

Note that prime denotes differentiation with respect to x, while ξ is more like a place-holding variable. So, we have replaced $f(x)$ by a delta function, in order to solve for the Green's function $g(x, \xi)$.

We have seen how to solve (49), (50) in the last section. The Green's function is

$$g(x, \xi) = \begin{cases} (1 - \xi)x & 0 < x < \xi \\ (1 - x)\xi & \xi < x < 1. \end{cases} \tag{51}$$

You can notice that this is exactly the solution (49) one would obtain via variation of parameters.

How to get back to the solution of (47), (48)? For each ξ, the Green's function gives the solution if a point heat source of unit strength were placed at $x = \xi$. Conceptually, then, to get the full solution we must "add up" the point sources, scaled by the value of the heat source at each point:

$$y(x) = \int_0^1 g(x, \xi) f(\xi)\, d\xi. \tag{52}$$

To verify that this is indeed a solution, we can plug (52) into (47):

$$-y''(x) = \int_0^1 -g''(x,\xi)f(\xi)\,dx = \int_0^1 \delta(x-\xi)f(\xi)\,dx = f(x)\ \checkmark \tag{53}$$

3.4 General linear BVP

We now consider a general nth order linear BVP with arbitrary continuous forcing function,

$$Ly(x) = a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = f(x) \tag{54}$$

for $a < x < b$, where each $a_i = a_i(x)$ is a continuous function, and moreover $a_n(x) \neq 0\ \forall x$[5]. Along with (54) are n boundary conditions, each a linear combination of y and derivatives up to $y^{(n-1)}$, evaluated at $x = a, b$. For instance, in the case $n = 2$, the general form is:

$$\begin{aligned} B_1 y &\equiv \alpha_{11}y(a) + \alpha_{12}y'(a) + \beta_{11}y(b) + \beta_{12}y'(b) = \gamma_1 \\ B_2 y &\equiv \alpha_{21}y(a) + \alpha_{22}y'(a) + \beta_{21}y(b) + \beta_{22}y'(b) = \gamma_2. \end{aligned} \tag{55}$$

3.5 General Green's Function

In the same way as in Section 3.3.4, to solve (54) with homogeneous BC

$$B_i y = 0, \quad i = 1 \ldots n-1,$$

we first determine the Green's function by solving

$$\begin{aligned} &Lg(x,\xi) = \delta(x-\xi), \quad a < x < b \\ &B_i g = 0. \end{aligned} \tag{56}$$

As before,

$$Lg(x,\xi) = \delta(x-\xi)$$

implies

$$Lg(x,\xi) = 0 \quad \text{on } a < x < \xi,\ \ \xi < x < b,$$

i.e. we have a homogeneous problem to solve on two separate domains. As before, we require extra conditions, which come by integrating $Lg(x,\xi) = \delta(x-\xi)$ across $x = \xi$:

$$\int_{\xi-}^{\xi+} a_n g^{(n)}(x,\xi) + \cdots + a_0 g(x,\xi)\,d\xi = \int_{\xi-}^{\xi+} \delta(x-\xi)\,d\xi. \tag{57}$$

[5] We'll return to the case where $a_n(x) = 0$ somewhere in the domain later in the course.

The right hand side clearly integrates to one. If we were to perform an integration by parts on the first term of the left hand side, we would obtain

$$a_n(x)g^{(n-1)}(x,\xi)]_{\xi-}^{\xi+} + \int_{\xi-}^{\xi+} (a_{n-1} - a_n')g^{(n-1)} + \cdots + a_0 g(x,\xi)\, d\xi = 1.$$

This equation is balanced by setting a jump condition on the $n-1$st derivative:

$$g^{(n-1)}(x,\xi)]_{\xi-}^{\xi+} = 1/a_n(\xi),$$

and taking all lower derivatives to be continuous across $x = \xi$:

$$g^{(j)}(x,\xi)]_{\xi-}^{\xi+} = 0, \quad j = 0, 1, \ldots n-2.$$

Once the Green's function is determined, the solution to the BVP is given by

$$y(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi. \tag{58}$$

3.6 Another view

There is one more way of viewing the GF. Start from $Ly(x) = f(x)$, and take an inner product with $G(x,\xi)$ on both sides of the equation[6]. We are not assuming we know G, rather we want to find properties it should satisfy for us to solve the equation. We obtain

$$\langle Ly, G\rangle = \langle G(x,\xi), f(x)\rangle = \int_a^b G(x,\xi) f(x)\, dx. \tag{59}$$

(Note the integration is over x). Now, using the adjoint, we can write

$$\langle Ly, G\rangle = \langle y, L^* G\rangle \tag{60}$$

The idea now is to isolate y. This can be accomplished if

$$L^* G(x,\xi) = \delta(x - \xi) \tag{61}$$

in which case the left hand side leaves just $y(\xi)$, and we have the solution

$$y(\xi) = \int_a^b G(x,\xi) f(x)\, dx. \tag{62}$$

[6] G will be the Green's function, but not quite the same one we've constructed, so I am differentiating by using capital G.

Comparing with our previous construction, here the big difference is that the GF is constructed through the adjoint operator – hence we will refer to this as the adjoint Green's function. Compare the form of solution with the form (30):

$$y(x) = \int_a^b g(x,t) f(t)\, dt, \tag{63}$$

we see the subtle difference that in (62) we integrate over the first variable of the adjoint GF, and the second variable of the GF. For a self-adjoint operator, the constructions are the same and we must get the same GF, and indeed as we've stated, the GF for a self-adjoint operator is symmetric.