2 Eigenfunction methods

Our first approach to solving linear inhomogeneous BVP's is through an eigenfunction expansion. The idea is to exploit the linearity of the operator by constructing a solution as a *superposition* of a (generally infinite) set of functions $\{y_i(x)\}$. In particular, the y_i will be the functions satisfying

$$Ly_i(x) = \lambda_i y_i(x), \tag{13}$$

along with homogeneous boundary conditions. Here y_i is an *eigenfunction* with corresponding *eigenvalue* λ_i. This is analogous to the linear algebra eigenproblem

$$\mathbf{A}\vec{x}_i = \lambda_i \vec{x}_i \tag{14}$$

where $\mathbf{A}$ is a matrix and $\vec{x}_i$ an eigenvector with eigenvalue λ_i.

2.1 Function spaces

In the same way as linear algebra utilises vector spaces, with linear differential operators we shall think of function spaces. Consider the infinite dimensional space of all reasonably well-behaved functions on the interval $a \leq x \leq b$.

Similar to a vector space, we can introduce a set of linearly independent basis functions $y_n(x), n = 1, 2, \ldots \infty$ such that any 'reasonable' function $f(x)$ can be written as a linear combination of these functions:

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x). \tag{15}$$

You have encountered this idea before with Fourier Series, where the basis functions are sines and cosines; this is merely a generalisation. Hence it should be clear that we can have different sets of basis functions.

We also define the **inner product**

$$\langle u, v \rangle = \int_a^b u(x)\overline{v(x)}\, dx. \tag{16}$$

Here the overbar denotes complex conjugate. In this course, we will rarely be concerned with complex valued functions. If it is clear that we are dealing with real functions, we may drop the overbar for simplicity.

2.1.1 Weighting function

In some instances, the eigenvalue problem and the inner product definition might include a given weighting function $\rho(x)$, required to be real and non-negative on $a \leq x \leq b$. In this case, the relations become

$$Ly_i(x) = \lambda_i \rho(x) y_i(x) \tag{17}$$

and

$$\langle u, v \rangle = \int_a^b \rho(x) u(x) \overline{v(x)}\, dx. \tag{18}$$

2.2 Adjoint

We also require the notion of the *adjoint* of an operator. For operator L with homogenous BC, the adjoint problem (L^* BC*) is defined by the inner product relation

$$\langle Ly, w \rangle = \langle y, L^* w \rangle. \tag{19}$$

To determine the adjoint, one needs to move the derivatives of the operator from y to w, and define adjoint boundary conditions so that all boundary terms vanish.

Example

Let $Ly = \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ with $a \leqslant x \leqslant b$, $y(a) = 0$ and $y'(b) - 3y(b) = 0$. We wish to find $L^* w$, such that

$$\int_a^b (w)(y'')\mathrm{d}x = \int_a^b (y)(L^* w)\mathrm{d}x$$

To do this, we need to shift the derivatives from y to w using integration by parts:

$$\begin{aligned} \int_a^b wy''\mathrm{d}x &= wy'|_a^b - \int_a^b w'y'\mathrm{d}x \\ &= wy' - w'y|_a^b + \int_a^b yw''\mathrm{d}x \end{aligned}$$

The integral gives the formal part so:

$$L^* w = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}.$$

The inner product only includes integral terms, so *the boundary terms must vanish*, which will define boundary conditions on w, i.e. this defines BC*. Here, we require

$$w(b)y'(b) - w'(b)y(b) - w(a)y'(a) + w'(a)y(a) = 0.$$

Using the BC's $y'(b) = 3y(b)$ and $y(a) = 0$, gives:

$$0 = y(b)\Big(3w(b) - w'(b)\Big) - w(a)y'(a) + \underbrace{w'(a)y(a)}_{=0}$$

As these terms need to vanish for all values of $y(b)$ and $y'(a)$, we can infer two boundary conditions on w:

- $y(b)$: $3w(b) - w'(b) = 0$
- $y'(a)$: $w(a) = 0$

If $L = L^*$ and $BC = BC^*$ then the problem is *self-adjoint*. If $L = L^*$ but $BC \neq BC^*$ we still call the operator self-adjoint. (Some books use the terminology *formally self-adjoint* if $L = L^*$ and *fully self-adjoint* if both $L = L^*$ and $BC = BC^*$).

2.2.1 Eigenfunction Properties

The main idea in solving the BVP is to construct a solution as a linear combination of eigenfunctions. There are two fundamental properties of eigenfunctions that will be vital to this approach.

1. **Eigenfunctions of the adjoint problem have the same eigenvalues as the original problem**

   That is,

$$Ly = \lambda y \Rightarrow \exists w \ni L^*w = \lambda w.$$

2. **Eigenfunctions corresponding to different eigenvalues are orthogonal**

   That is, if $Ly_j = \lambda_j y_j$ (so $L^* w_j = \lambda_j w_j$) and $Ly_k = \lambda_k y_k$ ($L^* w_k = \lambda_k w_k$), then for $\lambda_j \neq \lambda_k$, $\langle y_j, w_k \rangle = 0$.

   Proof

$$\begin{aligned} \lambda_j \langle y_j, w_k \rangle &= \langle \lambda_j y_j, w_k \rangle \\ &= \langle L y_j, w_k \rangle \\ &= \langle y_j, L^* w_k \rangle \\ &= \langle y_j, \lambda_k w_k \rangle \\ &= \lambda_k \langle y_j, w_k \rangle. \end{aligned}$$

   But $\lambda_j \neq \lambda_k$ so $\langle y_j, w_k \rangle = 0$. (The proof is exactly as for matrix problems.)

2.3 Inhomogeneous solution process

We are now in a position to outline the construction of solution to the BVP

$$Ly = f(x)$$

with linear, homogeneous, separated boundary conditions, denoted $BC_1(a) = 0$, $BC_2(b) = 0$.

Step 1: Solve the eigenvalue problem

$$Ly = \lambda y, \quad BC_1(a) = 0, \; BC_2(b) = 0$$

to obtain the eigenvalue-eigenfunction pairs (λ_j, y_j).

Step 2: Solve the adjoint eigenvalue problem

$$L^* w = \lambda w, \quad BC_1^*(a) = 0, \; BC_2^*(b) = 0$$

to obtain (λ_j, w_j).

Step 3: Assume a solution to the full system $Ly = f(x)$ of the form

$$y = \sum_i c_i y_i(x).$$

To determine the coefficients c_i, start from $Ly = f$ and take an inner product with w_k:

$$\begin{aligned} Ly &= f(x) \\ \Rightarrow \langle Ly, w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \langle y, L^* w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \langle y, \lambda_k w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \lambda_k \langle \sum_i c_i y_i, w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \lambda_k c_k \langle y_k, w_k \rangle &= \langle f, w_k \rangle \end{aligned} \tag{20}$$

We can solve the last equality for the c_k, and we are done! Note that in the last step we have used the orthogonality property $\langle y_j, w_k \rangle = 0, \quad j \neq k$.

2.4 Some simple solutions

Note that this construction requires that we are able to determine the eigenvalues/eigenfunctions in the first place. This is by no means guaranteed. But it will be useful to recall some simple cases, solvable using techniques you've learned before:

- Constant coefficients

$$Ly \equiv ay'' + by' + cy = \lambda y$$

Try $y = e^{mx}$, then:

$$am^2 + bm + (c - \lambda) = 0$$

Then:

1. Find roots m_i of the quadratic.
2. The general solution is: $y = A_1 e^{m_1 x} + A_2 e^{m_2 x}$. But note there are 3 unknowns: A_1, A_2, and λ, while for a second order equation there will only be two BC's.
3. Apply first BC to relate A_1 and A_2.
4. Apply second BC to determine values for λ.

- Cauchy-Euler

$$Ly \equiv ax^2 y'' + bxy' + cy = \lambda y$$

Try $y = x^m$, then:

$$am(m-1) + bm + (c - \lambda) = 0$$

Then $y = A_1 x^{m_1} + A_2 x^{m_2}$, and repeat the steps above.

2.5 A note on boundary conditions

In the above construction we assumed homogeneous boundary conditions. In the general case of an inhomogeneous system with inhomogenous boundary conditions,

$$\begin{aligned} Lu &= f(x) \\ B_i u &= \gamma_i \end{aligned} \tag{21}$$

a useful technique is to split the system in two, i.e. solve both

$$Lu_1 = f(x), \quad B_i u_1 = 0 \tag{22}$$

and

$$Lu_2 = 0, \quad B_i u_2 = \gamma_i. \tag{23}$$

Here, solving for $u_1(x)$ has the difficulty of the forcing function but with zero BC's while the other equation is homogeneous but has the non-zero BC's. Due to linearity, it is easy to see that $u(x) = u_1(x) + u_2(x)$ solves the full system (21).

This decomposition can always be performed[1] and since solving (23) tends to be an easier matter (for linear systems!), it is safe for us to primarily focus on the technique of solving the system (22), i.e. homogeneous boundary conditions.

[1] As we shall see in Section 5, it requires caution if there is a zero eigenvalue $\lambda = 0$.

For completeness it is worth noting that one *can* solve BVPs with inhomogeneous BC using an eigenfunction expansion and without needing a decomposition. The keys are:

1. The eigenfunctions are *always* determined using homogeneous boundary conditions. Thus, eigenfunctions won't change whether you "decompose" or not. The difference comes in:

2. In going from Line 2 to 3 of (20), care must be taken in the integration by parts, as boundary terms will generally still be present. (Can you see why?) These extra boundary terms then carry through to the formula for the c_k.

2.6 Example

Let $y'' = f(x)$ with $0 \leqslant x \leqslant 1$, $y(0) = \alpha$ and $y(1) = \beta$. Then:

BC's Incorporated Solution Route

1. Solve $y'' = \lambda y$, with $y(0) = 0$ and $y(1) = 0$.

   We get $y_k(x) = \sin(k\pi x)$ and $\lambda_k = -k^2\pi^2$ with $k = 1, 2, 3, \ldots$.

   The problem is self-adjoint (show this as an exercise), so $w_k = y_k = \sin(k\pi x)$ and $w_k'' = -\lambda_k w_k$.

2.

$$\begin{aligned}
y'' &= f(x) \\
\int_0^1 w_k y'' \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 + \int_0^1 w_k'' y \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 + \lambda_k \int_0^1 w_k y \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 + \lambda_k c_k \int_0^1 w_k y_k \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x \\
\Rightarrow (y'w_k - yw_k')|_0^1 - k^2\pi^2 c_k \int_0^1 \sin^2(k\pi x) \mathrm{d}x &= \int_0^1 w_k f \mathrm{d}x
\end{aligned}$$

3. Now $\int_0^1 \sin^2(k\pi x)\,dx = 1/2$, and $w_k = \sin(k\pi x)$, hence

$$y' w_k - y w_k'|_0^1 = -k\pi\cos(k\pi)y(1) + k\pi\cos(0)y(0)$$
$$\Rightarrow -\beta k\pi(-1)^k + \alpha k\pi - \frac{1}{2}k^2\pi^2 c_k = \int_0^1 f(x)\sin k\pi x \mathrm{d}x$$

Solving for c_k gives us $y(x)$ as a Fourier series.

Decomposed Solution Route

1. Solve two systems separately:

$$y'' = f(x), \quad y(0) = y(1) = 0$$
$$u'' = 0, \quad u(0) = \alpha,\ u(1) = \beta$$

2. To solve for y, since BC=0 we can jump straight to the formula

$$c_k = -\frac{\langle f, w_k\rangle}{\lambda_k\langle y_k, w_k\rangle} = -\frac{2\int_0^1 f(x)\sin(k\pi x)\,dx}{k^2\pi^2}.$$

3. The solution for u is easily obtained as

$$u = (\beta - \alpha)x + \alpha$$

4. The full solution is $y(x) + u(x)$.

Although they look different, both approaches give the same solution. Either way, we see that self-adjoint problems are great: they are less work since the w_k's are the same as the y_k's.

2.7 Connection with linear algebra

There are direct parallels between linear algebra and linear BVPs:

$$\begin{array}{ccc} \underline{\text{Linear algebra}} & & \underline{\text{Linear BVP}} \\ \text{vector } \vec{v} \in \mathbb{R}^n & \longleftrightarrow & \text{function } y(x) \text{ for } a \leqslant x \leqslant b \end{array}$$

$$\underbrace{\vec{v} \cdot \vec{w} = \sum_{k-1}^{n} v_k w_k}_{\text{dot product}} \quad \longleftrightarrow \quad \underbrace{\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\mathrm{d}x}_{\text{inner product}}$$

$$\underbrace{\| \vec{v} \|^2 = \vec{v} \cdot \vec{v} \geqslant 0}_{\text{norm}} \quad \longleftrightarrow \quad \underbrace{\| f \|^2 = \langle f, f \rangle \geqslant 0}_{\text{norm}}$$

$$\perp \text{ vector } \vec{v} \cdot \vec{w} = 0 \quad \longleftrightarrow \quad \text{orthogonal function } \langle f, g \rangle = 0$$

$$\text{Matrix } A \quad \longleftrightarrow \quad \text{Linear Differential Operator } L$$

Given a vector $\vec{v}$, then the product $A\vec{v}$ is a new vector. Similarly, given a function $y(x)$,

$$Ly = a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy$$

evaluates to a new function on $a \leqslant x \leqslant b$.
In linear algebra, a common problem is to solve the equation

$$A\vec{v} = \vec{b}$$

for $\vec{v}$, given matrix A and vector $\vec{b}$. Compare that to our general task of solving $Ly = f$ for y, given operator L and RHS f.

Eigenvalue problems

$$\begin{array}{ccc} \underline{\text{Linear algebra}} & & \underline{\text{Linear BVP}} \\ A\vec{v} = \lambda\vec{v} & \longleftrightarrow & Ly = \lambda y \end{array}$$

How many eigenvalues?

Linear algebra	Linear BVP
A is $n \times n$	L is order n
Solve $\|A - \lambda I\| = 0$	
$\Rightarrow n$ eigenvalues	∞ eigenvalues

Adjoint

	Linear algebra	Linear BVP
	$A \to A^T$	$L \to L^*$ BC's $\to$ BC*'s
Self adjoint if	$A = A^T$	$L = L^*$, BC=BC*

A self-adjoint matrix is called Hermitian. A self-adjoint linear differential operator is also referred to as Hermitian. We next look at a particular class of Hermitian operator – Sturm-Liouville operators – that occurs quite commonly and has very useful properties.

2.8 Sturm-Liouville theory

Sturm–Liouville (SL) theory of second order concerns self-adjoint operators of the form:

$$Ly = \lambda r(x) y$$

where $r(x) \geqslant 0$ is a *weighting function*, and the operator L is of the form

$$Ly = \frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y, \quad a \leq x \leq b \tag{24}$$

The functions p, q, and r are all assumed to be real. It is easy to check that the operator is formally self-adjoint. It is fully self-adjoint if the boundary conditions take the separated form

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\ \alpha_3 y(b) + \alpha_4 y'(b) &= 0. \end{aligned}$$

Observe also that if $p(a) = p(b) = 0$, then $\langle Ly, w\rangle = \langle y, Lw\rangle$ irrespective of boundary conditions. This defines the so-called natural interval $[a, b]$ for the problem.

2.8.1 Inhomogeneous SL problems

Since a SL operator is self-adjoint, the eigenfunction expansion process is quite straightforward. Consider

$$Ly = f(x)$$

with homogeneous BC's. The system can be solved with an eigenfunction expansion in the same manner as in Section 2.3:

$$\begin{aligned} Ly &= f(x) \\ \Rightarrow \langle Ly, y_k \rangle &= \langle f, y_k \rangle \\ \Rightarrow \langle y, Ly_k \rangle &= \langle f, y_k \rangle \qquad (\text{since } L^* = L, w_k = y_k) \\ \Rightarrow \langle y, \lambda_k r y_k \rangle &= \langle f, y_k \rangle \\ \Rightarrow \lambda_k c_k \langle y_k, r y_k \rangle &= \langle f, y_k \rangle. \end{aligned} \tag{25}$$

Thus we obtain the formula

$$c_k = \frac{\langle f, y_k \rangle}{\lambda_k \langle y_k, r y_k \rangle} \tag{26}$$

and the full solution is given by

$$y = \sum_k c_k y_k.$$

2.8.2 Transforming an operator to SL form

Many problems encountered in physical systems are Sturm-Liouville. In fact, though, any operator

$$Ly \equiv a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x)$$

with $a_2(x) \neq 0$ in the interval can be converted to a SL operator.
To *transform* to a self-adjoint SL operator, multiply by an integrating factor function $\mu(x)$:

$$\mu a_2(x)y''(x) + \mu a_1 y'(x) + \mu a_0 y$$

We then choose μ so that the first and second derivatives collapse, i.e. so it can be expressed in the form

$$\frac{d}{dx}(py') + qy$$

Suppose we are considering the problem

$$Ly = f(x)$$

where L is not Sturm-Liouville. We could solve following the approach in Eq'n (20); alternatively we could convert to Sturm-Liouville first, and then proceed using the nice properties of a self-adjoint operator. So, is the problem self-adjoint or isn't it?? The key observation is that we are no longer solving the same problem. We have transformed to a new operator

$$\hat{L}y = \frac{d}{dx}(py') + qy$$

which does not satisfy the same equation as the original, that is $Ly = f$ while $\hat{L}y = \mu f$. They are both valid, and must ultimately lead to the same answer in the end.

2.8.3 Further properties

Orthogonality.
Due to the presence of the weighting function, the *orthogonality* relation is

$$\int_a^b y_k(x) y_j(x) r(x) \mathrm{d}x = 0. \tag{27}$$

Eigenvalues.
The functions p, q, r are real, so $\overline{L} = L$. Thus, taking the conjugate of both sides of $Ly_k = \lambda_k y_k$ gives

$$\begin{aligned} & L\,\overline{y_k} = \overline{\lambda_k}\ r\ \overline{y_k} \\ \Rightarrow\ & \langle y_k, L\,\overline{y_k}\rangle = \overline{\lambda_k}\langle y_k,\ r\overline{y_k}\rangle \\ & \text{but } \langle y_k, L\,\overline{y_k}\rangle = \langle Ly_k,\ \overline{y_k}\rangle = \lambda_k\langle ry_k,\ \overline{y_k}\rangle = \lambda_k\langle y_k,\ r\overline{y_k}\rangle \end{aligned} \tag{28}$$

$$\Rightarrow\ \overline{\lambda_k} = \lambda_k$$

Thus, all eigenvalues are real.
Moreover, if $a \le x \le b$ is a finite domain, then λ's are <u>discrete and countable</u>:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots$$

with $\lim_{k\to\infty} \lambda_k = \infty$.

Eigenfunctions.

The $\{y_k\}$ are a *complete set*, that is all $h(x)$ with $\int h^2 r\,dx < \infty$ can be expanded as

$$h(x) = \sum c_k y_k(x).$$

Take an inner product with $r(x)y_j(x)$:

$$\langle ry_j, h\rangle = \langle ry_j, \sum c_k y_k\rangle = \sum c_k \langle ry_j, y_k\rangle = c_j \langle ry_j, y_j\rangle$$

$$\Rightarrow c_j = \frac{\int_a^b h(x)y_j(x)r(x)\,dx}{\int_a^b y_j^2(x)r(x)\,dx}$$

Note: I've used $h(x)$ to make clear that we're not talking about the solution to the BVP, rather we are expanding any function that is suitably bounded on the same domain.

2.8.4 Other tidbits

Regular Sturm-Liouville Problems. If the system satisfies all of the above and the additional conditions

- $p(x) > 0$ and $r(x) > 0$ on $a \le x \le b$.
- $q(x) \le 0$ on $a \le x \le b$.
- BCs have $\alpha_1\alpha_2 \le 0$ and $\alpha_3\alpha_4 \ge 0$,

then all $\boxed{\lambda_k \ge 0}$

Proof: Using $\langle y_k, Ly_k + \lambda_k r y_k\rangle = 0$,

$$\begin{aligned}
\int_a^b y(py')'\,dx + \int_a^b yqy\,dx + \int_a^b y\lambda ry\,dx &= 0 \\
\int_a^b y(py')'\,dx + \int_a^b qy^2\,dx + \lambda\int_a^b ry^2\,dx &= 0 \\
pyy'\Big|_a^b - \int_a^b p(y')^2\,dx + \int_a^b qy^2\,dx + \lambda\int_a^b ry^2\,dx &= 0
\end{aligned}$$

$$\lambda = \left[\int_a^b p(y')^2\,dx - \left(\int_a^b qy^2\,dx + pyy'\Big|_a^b\right)\right] \Big/ \int_a^b ry^2\,dx \ge 0$$

As a side note, the **Rayleigh quotient**, $R[y] = \langle y, Ly\rangle / \langle y, ry\rangle$, is used extensively in analysis.

Comparison theorem: If $\lambda_j > \lambda_k$ then the zeroes of $y_k(x)$ on $a < x < b$ lie between successive zeroes of $y_j(x)$.

Oscillation theorem [Simplest version]: The k^{th} eigenfunction will have k zeroes on $a < x < b$ $(k = 0, 1, 2, \cdots)$.

Monotonicity theorem: Comparing two SL problems, SL and $\widetilde{\text{SL}}$, with the same boundary conditions, the eigenvalues will satisfy $\tilde{\lambda}_k > \lambda_k$ if

$$\tilde{p}(x) > p(x) \quad \text{or} \quad \tilde{q}(x) > q(x) \quad \text{or} \quad \tilde{r}(x) < r(x) \quad \text{or} \quad (\tilde{a}, \tilde{b}) \subset (a, b)$$