1 Introduction

In this course, we will explore various techniques for solving differential equations, building on basic techniques for solving Differential Equations. A primary concern will be finding ways to solve and understand inhomogeneous linear boundary value problems (BVPs), that is an ordinary differential equation (ODE)

$$Lu(x) = f(x), \quad a < x < b \tag{1}$$

where L is a linear differential operator of the form

$$Lu = a_n u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \cdots + a_1 u'(x) + a_0 u(x) \tag{2}$$

and the function $f(x)$ on the right hand side (RHS) is a *forcing function* in the system. Along with equation (1) are boundary conditions at $x = a$ and $x = b$.

Some questions we will consider:

1. How do we solve the system for an arbitrary function $f(x)$?

2. Is there always a solution? If so, is it unique?

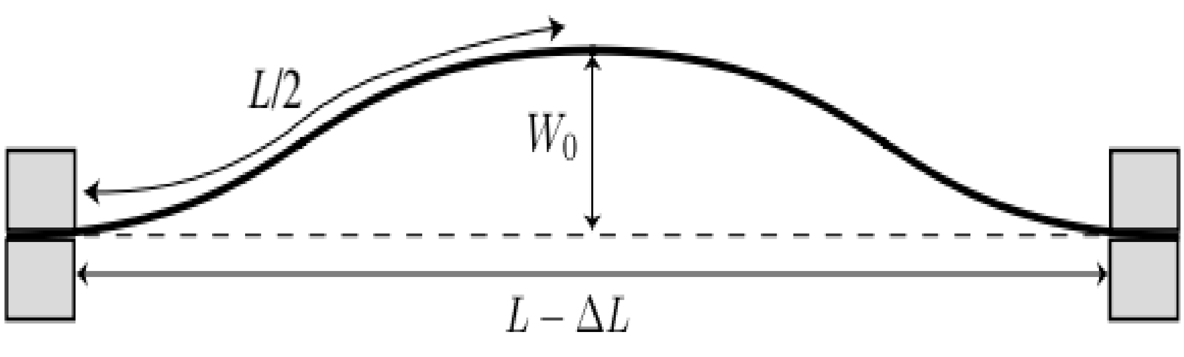

Figure 1: The notation used for the geometry of a beam of length L, which is clamped at its ends.

3. What is the effect of the boundary conditions?

4. Can we solve if the $a_k = a_k(x)$ are functions of x?

1.1 Motivating examples

Deflection of a beam: Consider the setup in figure 1. Under the assumptions of small slopes, the following beam equation describes the profile $w(x)$ of a beam in response to an applied force $F(x)$:

$$B\frac{\partial^4 w}{\partial x^4} + T\frac{\partial^2 w}{\partial x^2} = F(x). \tag{3}$$

Here T is the compressive force and B is the bending stiffness of the beam. For the fourth order equation, we should add 4 boundary conditions. For instance, clamped ends imply

$$w(\pm L/2) = w'(\pm L/2) = 0. \tag{4}$$

Note that T is generally an unknown constant. Hence an extra condition is required, this comes in the form of either a constitutive equation or geometrical constraint. For instance, if the beam is inextensible (i.e. can't be stretched) , we add the condition that the total arc length is fixed equal to L, the length of the beam. As a practical application, w might describe the shape of a tree growing at an angle and sagging under self-weight.

Diffusion: The diffusion equation models many phenomena. Let $u = u(x,t)$ denote a quantity that depends on one spatial variable x and time $t > 0$. This could be a chemical concentration, population, mass, energy, etc. The diffusion equation can be derived in a very intuitive way: we imagine a segment of space $[a, b]$. In this segment the total amount of "stuff" is

$$\int_a^b u(x,t)A\,dx \tag{5}$$

where A is the cross-sectional area. Now, we simply say that the rate of change of "stuff" in the segment is equal to the amount that leaves/enters through the ends plus any stuff that is added/taken away by the external world. The amount leaving the ends is the flux, denoted ϕ, and let $f(x,t)$ be a local source function – this is the rate u is created or destroyed at x at time t; note it could depend on u itself.

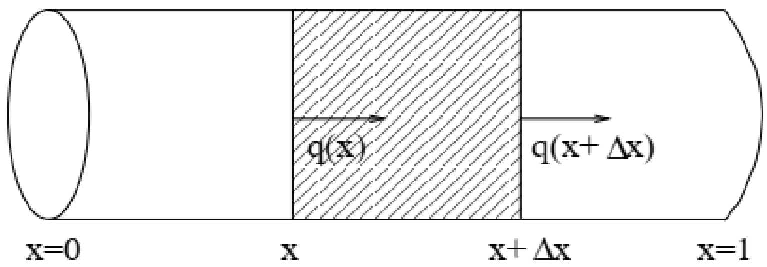

Figure 2: 1D diffusion

The balance is

$$\frac{d}{dt}\int_a^b u(x,t)\;dx = \phi(b,t) - \phi(a,t) + \int_a^b f(x,t,u)\;dx \tag{6}$$

By recognising

$$\phi(b,t) - \phi(a,t) = \int_a^b \phi_x(x,t)\;dx$$

we obtain

$$\int_a^b u_t + \phi_x - f\;dx = 0 \tag{7}$$

But this should hold for any segment, and thus the integrand must vanish:

$$u_t + \phi_x = f \tag{8}$$

To complete the system, the flux needs to be related to the quantity u. The simplest way is through Fick's Law, which states

$$\phi(x,t) = -Du_x(x,t) \tag{9}$$

where D is the diffusion constant. Combining (8) and (9), we obtain the diffusion equation

$$u_t - Du_{xx} = f. \tag{10}$$

The classic heat equation is the case $f = 0$. Many interesting and physically relevant situations are modelled with non-zero f. For example, (10) is a popular model for population dynamics, where f is used to capture growth and other interactions of the population.

Forced vibrations in wave equation: You've probably encountered the wave equation before:

$$u_{tt} = c^2 \Delta u, \tag{11}$$

where the constant c is the propagation speed in the medium, Δu is the Laplacian, equal to $u_{xx} + u_{yy}$ in 2D Cartesian coordinates. And you've probably seen in previous courses techniques for solving the wave equation in simple situations, e.g. motion of a 1D string. Things get more interesting when there is a forcing term involved, i.e.

$$u_{tt} - c^2 \Delta u = F(x, y), \tag{12}$$

For example, imagine a spider sitting on its web. If a fly gets caught in the web and struggles, the spider feels a vibration. But spiders have poor vision, and must determine where the fly is just from the vibrations. So the web is vibrating according to (12) with the fly creating a forcing term F, and the spider must "solve" the system to determine where the fly is. This is a trickier problem than you might think. One question to think about: how would you model the fly in this scenario?