Independent trials with two outcomes

In the preceding section we developed a way to determine a probability measure for any sequence of chance experiments where there are only a finite number of possibilities for each experiment. While this provides the framework for the general study of stochastic processes, it is too general to be studied in complete detail. Therefore, in probability theory we look for simplifying assumptions which will make our probability measure easier to work with. It is desired also that these assumptions be such as to apply to a variety of experiments which would occur in practice. In this book we shall limit outselves to the study of two types of processes. The first, the independent trials process, will be considered in the present section. This process was the first one to be studied extensively in probability theory. The second, the Markov chain process, is a process that is finding increasing application, particularly in the social and biological sciences, and will be considered in Section 4.13.

A process of independent trials applies to the following situation. Assume that there is a sequence of chance experiments, each of which consists of a repetition of a single experiment, carried out in such a way that the results of any one experiment in no way affect the results in any other experiment. We label the possible outcome of a single experiment by $a_1, \ldots, a_r$. We assume that we are also given probabilities $p_1, \ldots, p_r$ for each of these outcomes occurring on any single experiment, the probabilities being independent of previous results. The tree representing the possibilities for the sequence of experiments will have the same outcomes from each branch point, and the branch probabilities will be assigned by assigning probability p_j to any branch leading to outcome a_j. The tree measure determined in this way is the measure of an independent trials process. In this section we shall consider the important case of two outcomes for each experiment.

In the case of two outcomes we arbitrarily label one outcome "success" and the other "failure". For example, in repeated throws of a coin we might call heads success, and tails failure. We assume there is given a probability p for success and a probability $q = 1 - p$ for failure. The tree measure for a sequence of three such experiments is shown in Figure 4.13. The weights assigned to each path are indicated at the end of the path. The question which we now ask is the following. Given an independent trials process with two outcomes, what is the probability of exactly x successes in n experiments? We denote this probability by $f(n, x; p)$ to indicate that it depends upon n, x, and p.

Assume that we had a tree for this general situation, similar to the tree in Figure 4.13 for three experiments, with the branch points labeled S for success and F for failure. Then the truth set of the statement "Exactly x successes occur" consists of all paths which go through x branch points labeled S and $n - x$ labeled F. To find the probability of this statement we must add the weights for all such paths. We are helped first by the fact that our tree measure assigns the same weight to any such path, namely $p^x q^{n-x}$. The reason for this is that every branch leading to an S is assigned probability p, and every branch leading to F is assigned probability q, and in the product there will be x p's and $(n - x)$ q's. To find the desired probability we need only find the number of paths in the truth set of the statement "Exactly x

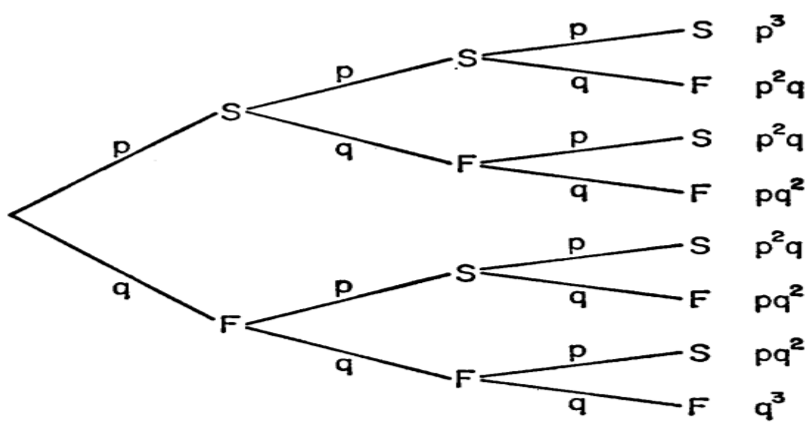

Figure 4.13: ◇

successes occur". To each such path we make correspond an ordered partition of the integers from 1 to n which has two cells, x elements in the first and $n - x$ in the second. We do this by putting the numbers of the experiments on which success occurred in the first cell and those for which failure occurred in the second cell. Since there are $\binom{n}{x}$ such partitions there are also this number of paths in the truth set of the statement considered. Thus we have proved:

In an independent trials process with two outcomes the probability of exactly x successes in n experiments is given by

$$f(n, x; p) = \binom{n}{x} p^x q^{n-x}.$$

Example 4.17 Consider n throws of an ordinary coin. We label heads "success" and tails "failure", and we assume that the probability is $\frac{1}{2}$ for heads on any one throw independently of the outcome of any other throw. Then the probability that exactly x heads will turn up is

$$f(n, x; \frac{1}{2}) = \binom{n}{x} (\frac{1}{2})^n.$$

For example, in 100 throws the probability that exactly 50 heads will

turn up is

$$f(100, 50; \frac{1}{2}) = \binom{100}{50} (\frac{1}{2})^{100},$$

which is approximately .08. Thus we see that it is quite unlikely that exactly one-half of the tosses will result in heads. On the other hand, suppose that we ask for the probability that nearly one-half of the tosses will be heads. To be more precise, let us ask for the probability that the number of heads which occur does not deviate by more than l0 from 50. To find this we must add $f(100, x; \frac{1}{2})$ for $x = 40, 41, \ldots, 60$. If this is done, we obtain a probability of approximately .96. Thus, while it is unlikely that exactly 50 heads will occur, it is very likely that the number of heads which occur will not deviate from 50 by more than l0. ◇

Example 4.18 Assume that we have a machine which, on the basis of data given, is to predict the outcome of an election as either a Republican victory or a Democratic victory. If two identical machines are given the same data, they should predict the same result. We assume, however, that any such machine has a certain probability q of reversing the prediction that it would ordinarily make, because of a mechanical or electrical failure. To improve the accuracy of our prediction we give the same data to r identical machines, and choose the answer which the majority of the machines give. To avoid ties we assume that r is odd. Let us see how this decreases the probability of an error due to a faulty machine.

Consider r experiments, where the jth experiment results in success if the jth machine produces the prediction which it would make when operating without any failure of parts. The probability of success is then $p = 1 - q$. The majority decision will agree with that of a perfectly operating machine if we have more than $r/2$ successes. Suppose, for example, that we have five machines, each of which has a probability of .1 of reversing the prediction because of a parts failure. Then the probability for success is .9, and the probability that the majority decision will be the desired one is

$$f(5, 3; 0.9) + f(5, 4; 0.9) + f(5, 5; 0.9)$$

which is found to be approximately .991 (see Exercise 3).

Thus the above procedure decreases the probability of error due to machine failure from .1 in the case of one machine to .009 for the case of five machines. ◇

Exercises

1. Compute for $n = 4$, $n = 8$, $n = 12$, and $n = 16$ the probability of obtaining exactly $\frac{1}{2}$ heads when an ordinary coin is thrown.

   [Ans. .375; .273; .226; .196.]

2. Compute for $n = 4$, $n = 8$, $n = 12$, and $n = 16$ the probability that the fraction of heads deviates from $\frac{1}{2}$ by less than $\frac{1}{5}$.

   [Ans. .375; .711, .854; .923.]

3. Verify that the probability .991 given in Example 4.18 is correct.

4. Assume that Peter and Paul match pennies four times. (In matching pennies, Peter wins a penny with probability $\frac{1}{2}$, and Paul wins a penny with probability $\frac{1}{2}$.) What is the probability that Peter wins more than Paul? Answer the same for five throws. For the case of 12,917 throws.

   [Ans. $\frac{5}{16}$; $\frac{1}{2}$; $\frac{1}{2}$.]

5. If an ordinary die is thrown four times, what is the probability that exactly two sixes will occur?

6. In a ten-question true-false exam, what is the probability of getting 70 per cent or better by guessing?

   [Ans. $\frac{11}{64}$.]

7. Assume that, every time a batter comes to bat, he or she has probability .3 for getting a hit. Assuming that hits form an independent trials process and that the batter comes to bat four times, what fraction of the games would he or she expect to get at least two hits? At least three hits? Four hits?

   [Ans. .348; .084; .008.]

8. A coin is to be thrown eight times. What is the most probable number of heads that will occur? What is the number having the highest probability, given that the first four throws resulted in heads?

9. A small factory has ten workers. The workers eat their lunch at one of two diners, and they are just as likely to eat in one as in the other. If the proprietors want to be more than .95 sure of having enough seats, how many seats must each of the diners have?

   [Ans. Eight seats.]

10. Suppose that five people are chosen at random and asked if they favor a certain proposal. If only 30 per cent of the people favor the proposal, what is the probability that a majority of the five people chosen will favor the proposal?

11. In Example 4.18, if the probability for a machine reversing its answer due to a parts failure is .2, how many machines would have to be used to make the probability greater than .89 that the answer obtained would be that which a machine with no failure would give?

    [Ans. Three machines.]

12. Assume that it is estimated that a torpedo will hit a ship with probability $\frac{1}{3}$. How many torpedos must be fired if it is desired that the probability for at least one hit should be greater than .9?

13. A student estimates that, if he or she takes four courses, he or she has probability .8 of passing each course. If he or she takes five courses, he or she has probability .7 of passing each course, and if he or she takes six courses he or she has probability .5 for passing each course. The student's only goal is to pass at least four courses. How many courses should he or she take for the best chance of achieving this goal?

    [Ans. 5.]

Supplementary exercises.

14. In a certain board game players move around the board, and each turn consists of a player's rolling a pair of dice. If a player is on the square Park Bench, he or she must roll a seven or doubles before being allowed to move out.

(a) What is the probability that a player stuck on Park Bench will be allowed to move out on the next turn?

[Ans. $\frac{1}{3}$.]

(b) How many times must a player stuck on Park Bench roll before the chances of getting out exceed $\frac{3}{4}$.

[Ans. 4.]

15. A restaurant orders five pieces of apple pie and five pieces of cherry pie. Assume that the restaurant has ten customers, and the probability that a customer will ask for apple pie is $\frac{3}{4}$ and for cherry pie is $\frac{1}{4}$.

    (a) What is the probability that the ten customers will all be able to have their first choice?

    (b) What number of each kind of pie should the restaurant order if it wishes to order ten pieces of pie and wants to maximize the probability that the ten customers will all have their first choice?

16. Show that it is more probable to get at least one ace with 4 dice than at least one double ace in 24 throws of two dice.

17. A thick coin, when tossed, will land "heads" with a probability of $\frac{5}{12}$, "tails" with a probability of $\frac{5}{12}$, and will land on edge with a probability of $\frac{1}{6}$. If it is tossed six times, what is the probability that it lands on edge exactly two times?

[Ans. .2009.]

18. Without actually computing the probabilities, find the value of x for which $f(20, x; .3)$ is largest.

19. A certain team has probability $\frac{2}{3}$ of winning whenever it plays.

    (a) What is the probability the team will win exactly four out of five games?

    [Ans. $\frac{80}{243}$.]

    (b) What is the probability the team will win at most four out of five games?

(c) What is the probability the team will win exactly four games out of five if it has already won the first two games of the five?

[Ans. $\frac{4}{9}$.]

A problem of decision

In the preceding sections we have dealt with the problem of calculating the probability of certain statements based on the assumption of a given probability measure. In a statistics problem, one is often called upon to make a decision in a case where the decision would be relatively easy to make if we could assign probabilities to certain statements, but we do not know how to assign these probabilities. For example, if a vaccine for a certain disease is proposed, we may be called upon to decide whether or not the vaccine should be used. We may decide that we could make the decision if we could compare the probability that a person vaccinated will get the disease with the probability that a person not vaccinated will get the disease. Statistical theory develops methods to obtain from experiments some information which will aid in estimating these probabilities, or will otherwise help in making the required decision. We shall illustrate a typical procedure.

Smith claims to have the ability to distinguish ale from beer and has bet Jones a dollar to that effect. Now Smith does not mean that he or she can distinguish beer from ale every single time, but rather a proportion of the time which is significantly greater than $\frac{1}{2}$.

Assume that it is possible to assign a number p which represents the probability that Smith can pick out the ale from a pair of glasses, one containing ale and one beer. We identify $p = \frac{1}{2}$ with having no ability, $p > \frac{1}{2}$ with having some ability, and $p < \frac{1}{2}$ with being able to distinguish, but having the wrong idea which is the ale. If we knew the value of p, we would award the dollar to Jones if p were $\leq \frac{1}{2}$, and to Smith if p were $> \frac{1}{2}$. As it stands, we have no knowledge of p and thus cannot make a decision. We perform an experiment and make a decision as follows.

Smith is given a pair of glasses, one containing ale and the other beer, and is asked to identify which is the ale. This procedure is repeated ten times, and the number of correct identifications is noted. If the number correct is at least eight, we award the dollar to Smith, and if it is less than eight, we award the dollar to Jones.

We now have a definite procedure and shall examine this procedure both from Jones's and Smith's points of view. We can make two kinds of errors. We may award the dollar to Smith when in fact the appropriate value of p is $\leq \frac{1}{2}$, or we may award the dollar to Jones when the appropriate value for p is $> \frac{1}{2}$ There is no way that these errors can be completely avoided. We hope that our procedure is such that each bettor will be convinced that, if he or she is right, he or she will very likely win the bet.

Jones believes that the true value of p is $\frac{1}{2}$. We shall calculate the probability of Jones winning the bet if this is indeed true. We assume that the individual tests are independent of each other and all have the same probability $\frac{1}{2}$ for success. (This assumption will be unreasonable if the glasses are too large.) We have then an independent trials process with $p = \frac{1}{2}$ to describe the entire experiment. The probability that Jones will win the bet is the probability that Smith gets fewer than eight correct. From the table in Figure 4.14 we compute that this probability is approximately .945. Thus Jones sees that, if he or she is right, it is very likely that he or she will win the bet.

Smith, on the other hand, believes that p is significantly greater than $\frac{1}{2}$. If Smith believes that p is as high as .9, we see from Figure 4.14 that the probability of Smith's getting eight or more correct is .930. Then both parties will be satisfied by the bet.

Suppose, however, that Smith thinks the value of p is only about .75. Then the probability that Smith will get eight or more correct and thus win the bet is .526. There is then only an approximately even chance that the experiment will discover Smith's abilities, and Smith probably will not be satisfied with this. If Smith really thinks his or her ability is represented by a p value of about $\frac{3}{4}$, we would have to devise a different method of awarding the dollar. We might, for example, propose that Smith win the bet if he or she gets seven or more correct. Then, if Smith has probability $\frac{3}{4}$ of being correct on a single trial, the probability that Smith will win the bet is approximately .776. If $p = \frac{1}{2}$ the probability that Jones will win the bet is about .828 under this new arrangement. Jones's chances of winning are thus decreased, but Smith may be able to convince him or her that it is a fairer arrangement than the first procedure.

In the above example, it was possible to make two kinds of errors. The probability of making these errors depended on the way we

Table of Values of $f(10, x; p)$

x \ p	0.1	0.25	0.50	0.75	0.90
0	.349	.056	.001	.000	.000
1	.387	.188	.010	.000	.000
2	.194	.282	.044	.000	.000
3	.057	.250	.117	.003	.000
4	.011	.146	.205	.016	.000
5	.001	.058	.246	.058	.001
6	.000	.016	.205	.146	.011
7	.000	.003	.117	.250	.057
8	.000	.000	.044	.282	.194
9	.000	.000	.010	.188	.387
10	.000	.000	.001	.056	.349

Figure 4.14: $\diamondsuit$

designed the experiment and the method we used for the required decision. In some cases we are not too worried about the errors and can make a relatively simple experiment. In other cases, errors are very important, and the experiment must be designed with that fact in mind. For example, the possibility of error is certainly important in the case that a vaccine for a given disease is proposed, and the statistician is asked to help in deciding whether or not it should be used. In this case it might be assumed that there is a certain probability p that a person will get the disease if not vaccinated, and a probability r that the person will get it if he or she is vaccinated. If we have some knowledge of the approximate value of p, we are then led to construct an experiment to decide whether r is greater than p, equal to p, or less than p.

The first case would be interpreted to mean that the vaccine actually tends to produce the disease, the second that it has no effect, and the third that it prevents the disease; so that we can make three kinds of errors. We could recommend acceptance when it is actually harmful, we could recommend acceptance when it has no effect, or finally we could reject it when it actually is effective. The first and third might result in the loss of lives, the second in the loss of time and money of those administering the test. Here it would certainly be important that the probability of the first and third kinds of errors be made small. To see how it is possible to make the probability of both errors small, we return to the case of Smith and Jones.

Suppose that, instead of demanding that Smith make at least eight correct identifications out of ten trials, we insist that Smith make at least 60 correct identifications out of 100 trials. (The glasses must now be very small.) Then, if $p = \frac{1}{2}$, the probability that Jones wins the bet is about .98; so that we are extremely unlikely to give the dollar to Smith when in fact it should go to Jones. (If $p < \frac{1}{2}$ it is even more likely that Jones will win.) If $p > \frac{1}{2}$ we can also calculate the probability that Smith will win the bet. These probabilities are shown in the graph in Figure 4.15. The dashed curve gives for comparison the corresponding probabilities for the test requiring eight out of ten correct. Note that with 100 trials, if p is $\frac{3}{4}$, the probability that Smith wins the bet is nearly 1, while in the case of eight out of ten, it was only about $\frac{1}{2}$. Thus in the case of 100 trials, it would be easy to convince both Smith and Jones that whichever one is correct is very likely to win the bet.

Thus we see that the probability of both types of errors can be made small at the expense of having a large number of experiments.

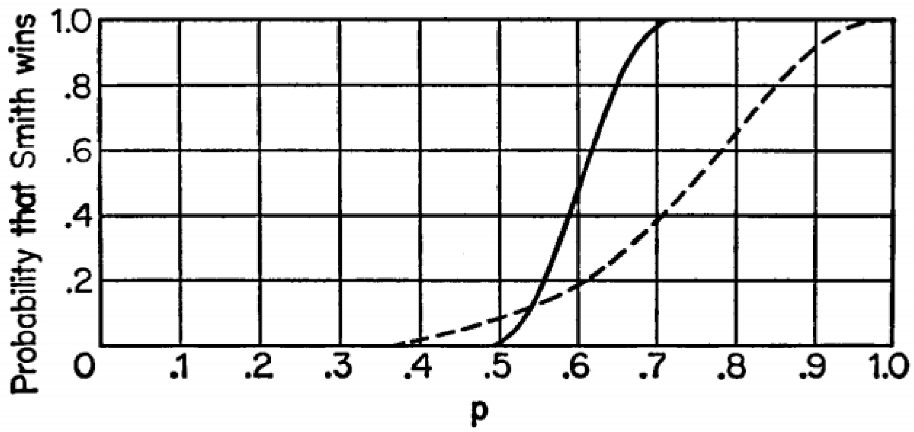

Figure 4.15: ♢

Exercises

1. Assume that in the beer and ale experiment Jones agrees to pay Smith if Smith gets at least nine out of ten correct.

   (a) What is the probability of Jones paying Smith even though Smith cannot distinguish beer and ale, and guesses?

   [Ans. .011.]

   (b) Suppose that Smith can distinguish with probability .9. What is the probability of not collecting from Jones?

   [Ans. .264.]

2. Suppose that in the beer and ale experiment Jones wishes the probability to be less than .1 that Smith will be paid if, in fact, Smith guesses. How many of ten trials must Jones insist that Smith get correct to achieve this?

3. In the analysis of the beer and ale experiment, we assume that the various trials were independent. Discuss several ways that error can enter, because of the nonindependence of the trials, and how this error can be eliminated. (For example, the glasses in which the beer and ale were served might be distinguishable.)

4. Consider the following two procedures for testing Smith's ability to distinguish beer from ale.

(a) Four glasses are given at each trial, three containing beer and one ale, and Smith is asked to pick out the one containing ale. This procedure is repeated ten times. Smith must guess correctly seven or more times. Find the probability that Smith wins by guessing.

[Ans. .003.]

(b) Ten glasses are given to Smith, and Smith is told that five contain beer and five ale, and asked to name the five that contain ale. Smith must choose all five correctly. Find the probability that Smith wins by guessing.

[Ans. .004.]

(c) Is there any reason to prefer one of these two tests over the other?

5. A testing service claims to have a method for predicting the order in which a group of freshmen will finish in their scholastic record at the end of college. The college agrees to try the method on a group of five students, and says that it will adopt the method if, for these five students, the prediction is either exactly correct or can be changed into the correct order by interchanging one pair of adjacent students in the predicted order. If the method is equivalent to simply guessing, what is the probability that it will be accepted?

[Ans. $\frac{1}{24}$.]

6. The standard treatment for a certain disease leads to a cure in $\frac{1}{4}$ of the cases. It is claimed that a new treatment will result in a cure in $\frac{3}{4}$ of the cases. The new treatment is to be tested on ten people having the disease. If seven or more are cured, the new treatment will be adopted. If three or fewer people are cured, the treatment will not be considered further. If the number cured is four, five, or six, the results will be called inconclusive, and a further study will be made. Find the probabilities for each of these three alternatives under the assumption first, that the new treatment has the same effectiveness as the old, and second, under the assumption that the claim made for the treatmnent is correct.

7. Three students debate the intelligence of Springer spaniels. One claims that Springers are mostly (say 90 per cent of them) intelligent. A second claims that very few (say 10 per cent) are intelligent, while a third one claims that a Springer is just as likely to be intelligent as not. They administer an intelligence test to ten Springers, classifying them as intelligent or not. They agree that the first student wins the bet if eight or more are intelligent, the second if two or fewer, the third in all other cases. For each student, calculate the probability that he or she wins the bet, if he or she is right.

[Ans. .930, .930, .890.]

8. Ten students take a test with ten problems. Each student on each question has probability $\frac{1}{2}$ of being right, if he or she does not cheat. The instructor determines the number of students who get each problem correct. If instructor finds on four or more problems there are fewer than three or more than seven correct, he or she considers this convincing evidence of communication between the students. Give a justification for the procedure. [Hint: The table in Figure 4.14 must be used twice, once for the probability of fewer than three or more than seven correct answers on a given problem, and the second time to find the probability of this happening on four or more problems.]