Bayes's probabilities

The following situation often occurs. Measures have been assigned in a possibility space $\mathcal{U}$. A complete set of alternatives, $p_1, p_2, \ldots, p_n$ has been singled out. Their probabilities are determined by the assigned measure. (Recall that a complete set of alternatives is a set of statements such that for any possible outcome one and only one of the statements is true.) We are now given that a statement q is true. We wish to compute the new probabilities for the alternatives relative to this information. That is, we wish the conditional probabilities $\Pr[p_j|q]$ for each p_j. We shall give two different methods for obtaining these probabilities.

The first is by a general formula. We illustrate this formula for the case of four alternatives: p_1, p_2, p_3, p_4. Consider $\Pr[p_2|q]$. From the definition of conditional probability,

$$\Pr[p_2|q] = \frac{\Pr[p_2 \wedge q]}{\Pr[q]}.$$

But since p_1, p_2, p_3, p_4 are a complete set of alternatives,

$$\Pr[q] = \Pr[p_1 \wedge q] + \Pr[p_2 \wedge q] + Pr[p_3 \wedge q] + \Pr[p_4 \wedge q].$$

Thus

$$\Pr[p_2|q] = \frac{\Pr[p_2 \wedge q]}{\Pr[p_1 \wedge q] + \Pr[p_2 \wedge q] + Pr[p_3 \wedge q] + \Pr[p_4 \wedge q]}.$$

Since $\Pr[p_j \wedge q] = \Pr[p_j]\Pr[q|p_j]$, we have the desired formula

$$\Pr[p_2|q] = \frac{\Pr[p_2]\Pr[q|p_2]}{\Pr[p_1]\Pr[q|p_1] + \Pr[p_2]\Pr[q|p_2] + \Pr[p_3]\Pr[q|p_3] + \Pr[p_4]\Pr[q|p_4]}.$$

Similar formulas apply for the other alternatives, and the formula generalizes in an obvious way to any number of alternatives. In its most general form it is called Bayes's theorem.

Example 4.14 Suppose that a freshman must choose among mathematics, physics, chemistry, and botany as his or her science course. On the basis of the interest he or she expressed, his or her adviser assigns probabilities of .4, .3, .2 and .1 to the student's choosing each of the four courses, respectively. The adviser does not hear which course the student actually chose, but at the end of the term the adviser hears that he or she received an A in the course chosen. On the basis of the difficulties of these courses the adviser estimates the probability of the student getting an A in mathematics to be .1, in physics .2, in chemistry .3, and in botany .9. How can the adviser revise the original estimates as to the probabilities of the student taking the various courses? Using Bayes's theorem we get

$$\Pr[\text{The student took math}|\text{The student got an A}] = \frac{(.4)(.1)}{(.4)(.1) + (.3)(.2) + (.2)(.3) + (.1)(.9)} = \frac{4}{25}$$

Similar computations assign probabilities of .24, .24, and .36 to the other three courses. Thus the new information, that the student received an A, had little effect on the probability of having taken physics or chemistry, but it has made mathematics less likely, and botany much more likely. ◇

It is important to note that knowing the conditional probabilities of q relative to the alternatives is not enough. Unless we also know the probabilities of the alternatives at the start, we cannot apply Bayes's theorem. However, in some situations it is reasonable to assume that the alternatives are equally probable at the start. In this case the factors $\Pr[p_1], \ldots, \Pr[p_4]$ cancel from our basic formula, and we get the special form of the theorem:

If $\Pr[p_1] = \Pr[p_2] = \Pr[p_3] = \Pr[p_4]$ then

$$\Pr[p_2|q] = \frac{\Pr[q|p_2]}{\Pr[q|p_1] + \Pr[q|p_2] + \Pr[q|p_3] + \Pr[q|p_4]}.$$

Example 4.15 In a sociological experiment the subjects are handed four sealed envelopes, each containing a problem. They are told to open one envelope and try to solve the problem in ten minutes. From past experience, the experimenter knows that the probability of their being able to solve the hardest problem is .1. With the other problems, they have probabilities of .3, .5, and .8. Assume the group succeeds within the allotted time. What is the probability that they selected the hardest problem? Since they have no way of knowing which problem is in which envelope, they choose at random, and we assign equal probabilities to the selection of the various problems. Hence the above simple formula applies. The probability of their having selected the hardest problem is

$$\frac{.1}{.1 + .3 + .5 + .8} = \frac{1}{17}.$$

◇

The second method of computing Bayes's probabilities is to draw a tree, and then to redraw the tree in a different order. This is illustrated in the following example.

Example 4.16 There are three urns. Each urn contains one white ball. In addition, urn I contains one black ball, urn II contains two, and urn III contains 3. An urn is selected and one ball is drawn. The probability for selecting the three urns is $\frac{1}{6}$, $\frac{1}{2}$, and $\frac{1}{3}$, respectively. If we know that a white ball is drawn, how does this alter the probability that a given urn was selected?

First we construct the ordinary tree and tree measure, in Figure 4.7.

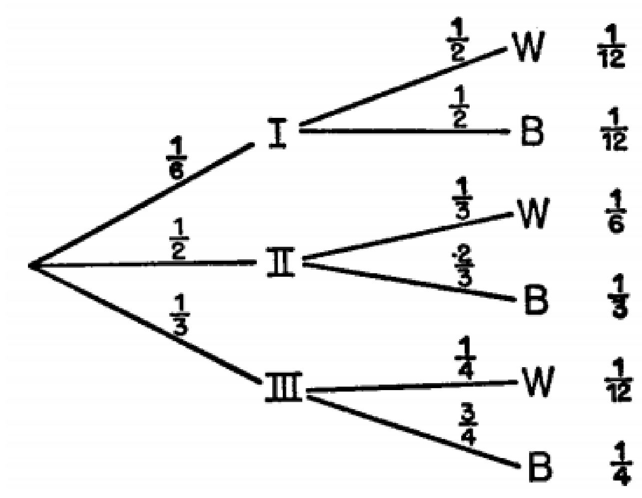

Figure 4.7: $\diamondsuit$

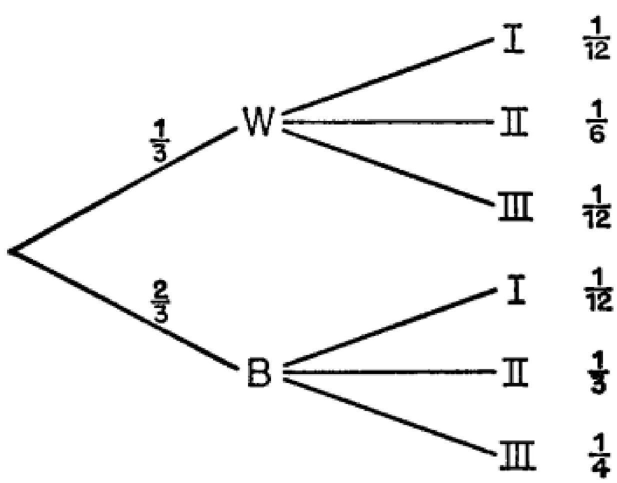

Figure 4.8: $\diamondsuit$

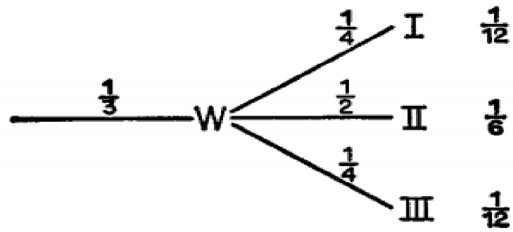

Figure 4.9: ◊

Next we redraw the tree, using the ball drawn as stage 1, and the urn selected as stage 2. (See Figure 4.8.) We have the same paths as before, but in a different order. So the path weights are read off from the previous tree. The probability of drawing a white ball is

$$\frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}.$$

This leaves the branch weights of the second stage to be computed. But this is simply a matter of division. For example, the branch weights for the branches starting at W must be $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ to yield the correct path weights. Thus, if a white ball is drawn, the probability of having selected urn I has increased to $\frac{1}{4}$, the probability of having picked urn III has fallen to $\frac{1}{4}$, while the probability of having chosen urn II is unchanged (see Figure 4.9). ◊

This method is particularly useful when we wish to compute all the conditional probabilities. We will apply the method next to Example 4.14. The tree and tree measure for this example in the natural order is shown in Figure 4.10. In that figure the letters M, P, C, and B stand for mathematics, physics, chemistry, and botany, respectively.

The tree drawn in reverse order is shown in Figure 4.11. Each path in this tree corresponds to one of the paths in the original tree. Therefore the path weights for this new tree are the same as the weights assigned to the corresponding paths in the first tree. The two branch weights at the first level represent the probability that the student receives an A or that he or she does not receive an A. These probabilities are also easily obtained from the first tree. In fact,

$$\Pr[A] = .04 + .06 + .06 + .09 = .25$$

and

$$\Pr[\neg A] = 1 - \Pr[A] = .75.$$

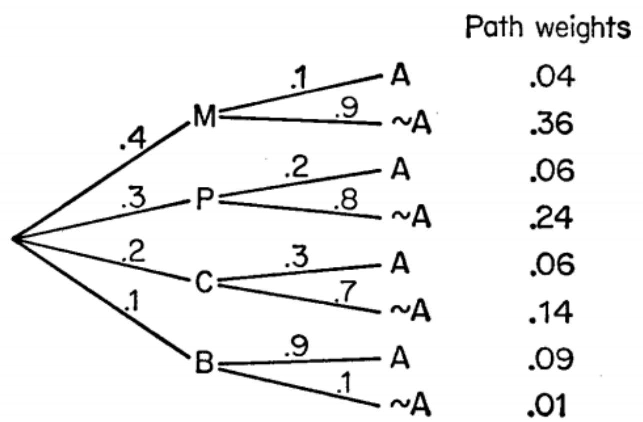

Figure 4.10: ♢

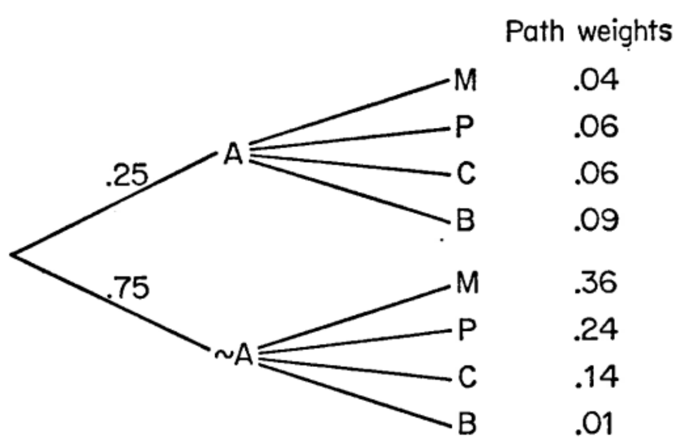

Figure 4.11: ♢

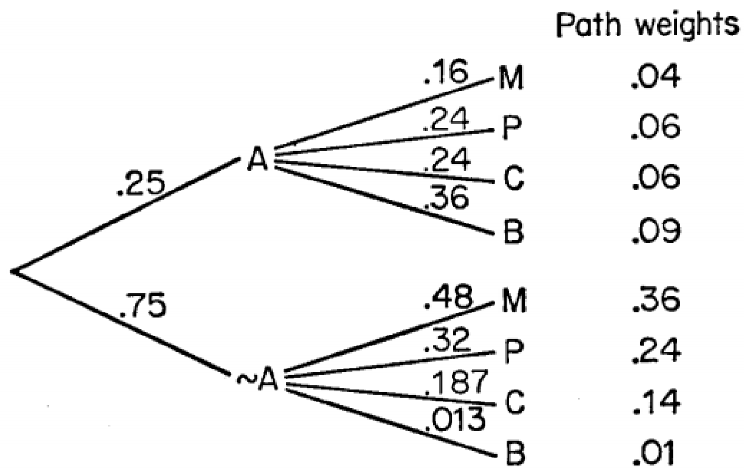

Figure 4.12: $\diamondsuit$

We have now enough information to obtain the branch weights at the second level, since the product of the branch weights must be the path weights. For example, to obtain $p_{\mathrm{A,M}}$ we have

$$.25 \cdot p_{\mathrm{A,M}} = .04;\ p_{\mathrm{A,M}} = .16.$$

But $p_{\mathrm{A,M}}$ is also the conditional probability that the student took math given that he or she got an A. Hence this is one of the new probabilities for the alternatives in the event that the student received an A. The other branch probabilities are found in the same way and represent the probabilities for the other alternatives. By this method we obtain the new probabilities for all alternatives under the hypothesis that the student receives an A as well as the hypothesis that the student does not receive an A. The results are shown in the completed tree in Figure 4.12.

Exercises

1. Urn I contains 7 red and 3 black balls and urn II contains 6 red and 4 black balls. An urn is chosen at random and two balls are drawn from it in succession without replacement. The first ball is red and the second black. Show that it is more probable that urn II was chosen than urn I.

2. A gambler is told that one of three slot machines pays off with probability $\frac{1}{2}$, while each of the other two pays off with probability $\frac{1}{3}$.

(a) If the gambler selects one at random and plays it twice, what is the probability that he or she will lose the first time and win the second?

[Ans. $\frac{25}{108}$.]

(b) If the gambler loses the first time and wins the second, what is the probability he or she chose the favorable machine?

[Ans. $\frac{9}{25}$.]

3. During the month of May the probability of a rainy day is .2. The Dodgers win on a clear day with probability .7, but on a rainy day only with probability .4. If we know that they won a certain game in May, what is the probability that it rained on that day?

[Ans. $\frac{1}{8}$.]

4. Construct a diagram to represent the truth sets of various statements occurring in the previous exercise.

5. On a multiple-choice exam there are four possible answers for each question. Therefore, if a student knows the right answer, he or she has probability 1 of choosing correctly; if the student is guessing, he or she has probability $\frac{1}{4}$ of choosing correctly. Let us further assume that a good student will know 90 per cent of the answers, a poor student only 50 per cent. If a good student has the right answer, what is the probability that he or she was only guessing? Answer the same question about a poor student, if the poor student has the right answer.

[Ans. $\frac{1}{37}$; $\frac{1}{5}$.]

6. Three economic theories are proposed at a given time, which appear to be equally likely on the basis of existing evidence. The state of the American economy is observed the following year, and it turns out that its actual development had probability .6 of happening according to the first theory; and probabilities .4 and .2 according to the others. How does this modify the probabilities of correctness of the three theories?

7. Let p_1, p_2, p_3, and p_4 be a set of equally likely alternatives. Let $\Pr[q|p_1] = a$, $\Pr[q|p_2] = b$, $\Pr[q|p_3] = c$, $\Pr[q|p_4] = d$. Show that if $a+b+c+d = 1$, then the revised probabilities of the alternatives relative to q are a, b, c, and d, respectively.

8. In poker, Smith holds a very strong hand and bets a considerable amount. The probability that Smith's opponent, Jones, has a better hand is .05. With a better hand Jones would raise the bet with probability .9, but with a poorer hand Jones would raise only with probability .2. Suppose that Jones raises, what is the new probability that he or she has a winning hand?

[Ans. $\frac{9}{47}$.]

9. A rat is allowed to choose one of five mazes at random. If we know that the probabilities of his or her getting through the various mazes in three minutes are .6, .3, .2, .1, .1, and we find that the rat escapes in three minutes, how probable is it that he or she chose the first maze? The second maze?

[Ans. $\frac{6}{13}$; $\frac{3}{13}$.]

10. Three men, A, B, and C, are in jail, and one of them is to be hanged the next day. The jailer knows which man will hang, but must not announce it. Man A says to the jailer, "Tell me the name of one of the other two who will not hang. If both are to go free, just toss a coin to decide which to say. Since I already know that at least one of them will go free, you are not giving away the secret." The jailer thinks a moment and then says, "No, this would not be fair to you. Right now you think the probability that you will hang is $\frac{1}{3}$, but if I tell you the name of one of the others who is to go free, your probability of hanging increases to $\frac{1}{2}$. You would not sleep as well tonight." Was the jailer's reasoning correct?

[Ans. No.]

11. One coin in a collection of 8 million coins has two heads. The rest are fair coins. A coin chosen at random from the collection is tossed ten times and comes up heads every time. What is the probability that it is the two-headed coin?