Voting power

We will develop a numerical measure of voting power that was suggested by L. S. Shapley and M. Shubik. While the measure will be explained in detail below, for the reasons for choosing this particular measure the reader is referred to the original paper.

First of all we must realize that the number of votes a member controls is not in itself a good measure of his or her power. If x has three votes and y has one vote, it does not necessarily follow that x has three times the power that y has. Thus if the committee has just three members $\{x, y, z\}$ and z also has only one vote, then x is a dictator and y is powerless.

The basic idea of the power index is found in considering various alignments of the committee members on a number of issues. The n members are ordered $x_1, x_2, \ldots, x_n$ according to how likely they are to vote for the measure. If the measure is to carry, we must persuade x_1 and x_2 up to x_i to vote for it until we have a winning coalition. If $\{x_1, x_2, \ldots, x_i\}$ is a winning coalition but $\{x_1, x_2, \ldots, x_{i-1}\}$ is not winning, then x_i is the crucial member of the coalition. We must persuade him or her to vote for the measure, and he or she is the one hardest to persuade of the i necessary members. We call x_i the pivot.

For a purely mathematical measure of the power of a member we do not consider the views of the members. Rather we consider all possible ways that the members could be aligned on an issue, and see how often a given member would be the pivot. That means considering all permutations, and there will be $n!$ of them. In each permutation one member will be the pivot. The frequency with which a member is the pivot of an alignment is a good measure of his or her voting power.

Definition. The *voting power* of a member of a committee is the number of alignments in which he or she is pivotal divided by the total number of alignments. (The total number of alignments, of course, is $n!$ for a committee of n members.)

Example 3.17 If all n members have one vote each, and it takes a majority vote to carry a measure, it is easy to see (by symmetry) that each member is pivot in $1/n$ of the alignments. Hence each member has power equal to $1/n$. Let us illustrate this for $n = 3$. There are $3! = 6$ alignments. It takes two votes to carry a measure; hence the second member is always the pivot. The alignments are: 1**2**3, 1**3**2, 2**1**3, 2**3**1, 3**1**2, 3**2**1. The pivots are emphasized. Each member is pivot twice, hence has power $\frac{2}{6} = \frac{1}{3}$. ◊

Example 3.18 Reconsider Example 2.9 of Section 2.6 from this point of view. There are 24 permutations of the four members. We will list them, with the pivot emphasized:

wxy**z**	wx**z**y	wyx**z**	wy**z**x	w**z**xy	w**z**yx
xwy**z**	xw**z**y	xyw**z**	xy**z**w	xz**w**y	xz**y**w
yxw**z**	yx**z**w	ywx**z**	yw**z**x	yz**x**w	yz**w**x
zx**y**w	zx**w**y	zy**x**w	zy**w**x	z**w**xy	z**w**yx

We see that z has power of $\frac{14}{24}$, w has $\frac{6}{24}$, x and y have $\frac{2}{24}$ each. (Or, simplified, they have $\frac{7}{12}$, $\frac{3}{12}$, $\frac{1}{12}$, $\frac{1}{12}$ power, respectively.) We note that these ratios are much further apart than the ratio of votes which is $3 : 2 : 1 : 1$. Here three votes are worth seven times as much as the single vote and more than twice as much as two votes. ◊

Example 3.19 Reconsider Example 2.10 of Section 2.6. By an analysis similar to the ones used so far it can be shown that in the Security Council of the United Nations before 1966, each of the Big Five had $\frac{76}{385}$ or approximately .197 power, while each of the small nations had approximately .002 power. (See Exercise 12.) This reproduces our intuitive feeling that, while the small nations in the Security Council are not powerless, nearly all the power is in the hands of the Big Five.

Example 3.20 In a committee of five each member has one vote, but the chair has veto power. Hence the minimal winning coalitions are three-member coalitions including the chair. There are $5! = 120$ permutations. The pivot cannot come before the chair, since without the chair we do not have a winning coalition. Hence, when the chair is in place number 3, 4, or 5, he or she is the pivot. This happens in $\frac{3}{5}$ of the permutations. When he or she is in position 1 or 2, then the number 3 member is pivot. The number of permutations in which the chair is in one of the first two posltions and a given member is third is $2 \cdot 3! = 12$. Hence the chair has power $\frac{3}{5}$, and each of the others has power $\frac{1}{10}$. ♢

Exercises

1. A committee of three makes decisions by majority vote. Write out all permutations, and calculate the voting powers if the three members have

   (a) One vote each.

   [Ans. $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$.]

   (b) One vote for two of them, two votes for the third.

   [Ans. $\frac{1}{6}, \frac{1}{6}, \frac{2}{3}$.]

   (c) One vote for two of them, three votes for the third.

   [Ans. $0, 0, 1$.]

   (d) One, two, and three votes, respectively.

   [Ans. $\frac{1}{6}, \frac{1}{6}, \frac{2}{3}$.]

   (e) Two votes each for two of them, and three votes for the third.

   [Ans. $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$.]

2. Prove that in any decision-making body the sum of the powers of the members is 1.

3. What is the power of a dictator? What is the power of a "powerless" member? Prove that your answers are correct.

4. A large company issued 100,000 shares. These are held by three stockholders, who have 50,000, 49,999, and one share, respectively. Calculate the powers of the three members.

[Ans. $\frac{2}{3}, \frac{1}{6}, \frac{1}{6}$.]

5. A committee consists of 100 members having one vote each, plus a chairman who can break ties. Calculate the power distribution. (Do not try to write out all permutations!)

6. In Exercise 5, give the chairman a veto instead of the power to break ties. How does this change the power distribution?

[Ans. The chairman has power $\frac{50}{101}$.]

7. How are the powers in Exercise 1 changed if the committee requires a $\frac{3}{4}$ vote to carry a measure?

8. If in a committee of five, requiring majority decisions, each member has one vote, then each has power $\frac{1}{5}$. Now let us suppose that two members team up, and always vote the same way. Does this increase their power? (The best way to represent this situation is by allowing only those permutations in which these two members are next to each other.)

[Ans. Yes, the pair's power increases from .4 to .5.]

9. If the minimal winning coalitions are known, show that the power of each member can be determined without knowing anything about the number of votes that each member controls.

10. Answer the following questions for a three-man committee.

    (a) Find all possible sets of minimal winning coalitions.

    (b) For each set of minimal winning coalitions find the distribution of voting power.

    (c) Verify that the various distributions of power found in Exercises 1 and 7 are the only ones possible.

11. In Exercise 1 parts 1a and 1e have the same answer, and parts 1b and 1d and Exercise 4 also have the same answer. Use the results of Exercise 9 to find a reason for these coincidences.

12. Compute the voting power of one of the Big Five in the Security Council of the United Nations as follows:

    (a) Show that for the nation to be pivotal it must be in the number 7 spot or later.

    (b) Show that there are $\binom{6}{2}6!4!$ permutations in which the nation is pivotal in the number 7 spot.

    (c) Find similar formulas for the number of permutations in which it is pivotal in the number 8, 9, 10, or 11 spot.

    (d) Use this information to find the total number of permutations in which it is pivotal, and from this compute the power of the nation.

13. Apply the method of Exercise 12 to the revised voting scheme in the Security Council (10 small-nation members, and 9 votes required to carry a measure). What is the power of a large nation? Has the power of one of the small nations increased or decreased?

    [Ans. $\frac{421}{2145}$ (nearly the same as before); decreased.]

Techniques for counting

We know that there is no single method or formula for solving all counting problems. There are, however, some useful techniques that can be learned. In this section we shall discuss two problems that illustrate important techniques.

The first problem illustrates the importance of looking for a general pattern in the examination of special cases. We have seen in Section 3.2 and Exercise 6 of that section, that the following formulas hold for the number of elements in the union of two and three sets, respectively.

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2),$$

$$\begin{aligned} n(A_1 \cup A_2 \cup A_3) &= n(A_1) + n(A_2) + n(A_3) \\ &\quad -n(A_1 \cap A_2) - n(A_1 \cap A_3) - n(A_2 \cap A_3) \\ &\quad +n(A_1 \cap A_2 \cap A_3). \end{aligned}$$

On the basis of these formulas we might conjecture that the number of elements in the union of any finite number of sets could be obtained by adding the numbers in each of the sets, then subtracting the numbers in each possible intersection of two sets, then adding the numbers in each possible intersection of three sets, etc. If this is correct, the formula for the intersection of four sets should be

$$\begin{aligned} n(A_1 \cup A_2 \cup A_3 \cup A_4) &= n(A_1) + n(A_2) + n(A_3) + n(A_4) \\ &\quad - n(A_1 \cap A_2) - n(A_1 \cap A_3) - n(A_1 \cap A_4) \\ &\quad - n(A_2 \cap A_3) - n(A_2 \cap A_4) - n(A_3 \cap A_4) \\ &\quad + n(A_1 \cap A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_4) \\ &\quad + n(A_1 \cap A_3 \cap A_4) + n(A_2 \cap A_3 \cap A_4) \\ &\quad - n(A_1 \cap A_2 \cap A_3 \cap A_4) \end{aligned} \tag{3.1}$$

Let us try to establish this formula. We must show that if u is an element of at least one of the four sets, then it is counted exactly once on the right-hand side of 3.1. We consider separately the cases where u is in exactly 1 of the sets, exactly 2 of the sets, etc.

For instance, if u is in exactly two of the sets it will be counted twice in the terms of the right-hand side of 3.1 that involve single sets, once in the terms that involve the intersection of two sets, and not at all in the terms that involve the intersections of three or four sets. Again, if u is in exactly three of the sets it will be counted three times in the terms involving single sets, twice in the terms involving intersections of two sets, once in the terms involving the intersections of three sets, and not at all in the last term involving the intersection of all four sets. Considering each possibility we have the following table.

Number of sets that contain u	Number of times it is counted
1	1
2	$2-1$
3	$3-3+1$
4	$4-6+4-1$

We see from this that, in every case, u is counted exactly once on the right-hand side of 3.1. Furthermore, if we look closely, we detect a pattern in the numbers in the righthand column of the above table. If we put a -1 in front of these numbers we have

1	$-1+1$
2	$-1+2-1$
3	$-1+3-3+1$
4	$-1+4-6+4-1$

We now recognize that these numbers are simply the numbers in the first four rows of the Pascal triangle, but with alternating + and − signs. Since we put a -1 in each row of the table, we want to show that the sum of each row is 0. If that is true, it should be a general property of the Pascal triangle. That is, if we put alternating signs in the jth row of the Pascal triangle, we should get a sum of 0. But this is indeed the case, since, by the binomial theorem, for $j>0$,

$$\begin{aligned} 0 &= \pm(1-1)^j \\ &= 1-\binom{j}{1}+\binom{j}{2}-\binom{j}{3}+\ldots\pm 1 \\ &= -1+\binom{j}{1}-\binom{j}{2}+\binom{j}{3}-\ldots\mp 1. \end{aligned}$$

Thus we have not only seen why the formula works for the case of four sets, but we have also found the method for proving the formula for the general case. That is, suppose we wish to establish that the number of elements in the union of n sets may be obtained as an alternating sum by adding the numbers of elements in each of the sets, subtracting the numbers of elements in each pairwise intersection of the sets, adding the numbers of elements in each intersection of three sets, etc. Consider an element u that is in exactly j of the sets. Let us see how many times u will be counted in the alternating sum. If it is in j of the sets, it will first be counted j times in the sum of the elements in the sets by themselves. For u to be in the intersection of two sets, we must choose two of the j sets to which it belongs. This can be done in $\binom{j}{2}$

different ways. Hence an amount $\binom{j}{2}$ will be subtracted from the sum. To be in the intersection of three sets, we must choose three of the j sets containing u. This can be done in $\binom{j}{3}$ different ways. Thus, an amount $\binom{j}{3}$ will be added to the sum, etc. Hence the total number of times u will be counted by the alternating sum is

$$\binom{j}{1} - \binom{j}{2} + \binom{j}{3} - \ldots \pm 1$$

since we have just seen that, if we add -1 to the sum, we obtain 0. Hence the sum itself must always be 1. That is, no matter how many sets u is in, it will be counted exactly once by the alternating sum, and this is true for every element u in the union. We have thus established the general formula

$$\begin{aligned} n(A_1 \cup A_2 \cup \ldots \cup A_n) &= n(A_1) + n(A_2) + \ldots + n(A_n) \\ &\quad - n(A_1 \cap A_2) - n(A_1 \cap A_3) - \ldots \\ &\quad + n(A_1 \cap A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_4) + \ldots \\ &\quad - \ldots \\ &\quad \pm n(A_1 \cap A_2 \cap \ldots \cap A_n). \end{aligned} \tag{3.2}$$

This formula is called the *inclusion-exclusion formula* for the number of elements in the union of n sets. It can be extended to formulas for counting the number of elements that occur in two of the sets, three of the sets, etc. See Exercises 21, 25, and 27.

Example 3.21 In a high school the following language enrollments are recorded for the senior class.

English	150
French	75
German	35
Spanish	50

Also, the following overlaps are noted.

Taking English and French	70
Taking English and German	30
Taking English and Spanish	40
Taking French and German	5
Taking English, French and German	2

If every student takes at least one language, how many seniors are there?

Let E, F, G, and S be the sets of students taking English, French, German, and Spanish, respectively. Using formula 3.1 and ignoring empty sets, we have

$$\begin{aligned} n(E \cup F \cup G \cup S) &= n(E) + n(F) + n(G) + n(S) \\ &\quad - n(E \cap F) - n(E \cap G) - n(E \cap S) - n(F \cap G) \\ &\quad + n(E \cap F \cap G) \\ &= 150 + 75 + 35 + 50 - 70 - 30 - 40 - 5 + 2 \\ &= 167. \end{aligned}$$

Since every student takes at least one language, the total number of students is 167. ◇

Example 3.22 The four words

TABLE, BASIN, CLASP, BLUSH

have the following interesting properties. Each word consists of five different letters. Any two words have exactly two letters in common. Any three words have one letter in common. No letter occurs in all four words. How many different letters are there?

Letting the words be sets of letters, there are $\binom{4}{1}$ ways of taking these sets one at a time, $\binom{4}{2}$ ways of taking them two at a time, etc. Hence formula 3.2 gives

$$\binom{4}{1} \cdot 5 - \binom{4}{2} \cdot 2 + \binom{4}{3} \cdot 1 - \binom{4}{4} \cdot 0 = 12$$

as the number of distinct letters. The reader should verify this answer by direct count. ◇

It often happens that a counting problem can be formulated in a number of different ways that sound quite different but that are in fact equivalent. And in one of these ways the answer may suggest itself readily. To illustrate how a reformulation can make a hard sounding problem easy, we give an alternate method for solving the problem considered in Exercise 13.

The problem is to count the number of ways that n indistinguishable objects can be put into r cells. For instance, if there are three objects and three cells, the number of different ways can be enumerated as follows (Using $\circ$ for object and bars to indicate the sides of the cells:

$\circ\circ\circ$		
$\circ\circ$	$\circ$	
$\circ\circ$		$\circ$
$\circ$	$\circ\circ$	
$\circ$	$\circ$	$\circ$
	$\circ\circ\circ$	
	$\circ\circ$	$\circ$
	$\circ$	$\circ\circ$
		$\circ\circ\circ$

We see that in this case there are ten ways the task can be accomplished. But the answer for the general case is not clear.

If we look at the problem in a slightly different manner, the answer suggests itself. Instead of putting the objects in the cells, we imagine putting the cells around the objects. In the above case we see that three cells are constructed from four bars. Two of these bars must be placed at the ends. The two other bars together with the three objects we regard as occupying five intermediate positions. Of these five intermediate positions we must choose two of them for bars and three for the objects. Hence the total number of ways we can accomplish the task is $\binom{5}{2} = \binom{5}{3} = 10$, which is the answer we got by counting all the ways.

For the general case we can argue in the same manner. We have r cells and n objects. We need $r+1$ bars to form the r cells, but two of these must be fixed on the ends. The remaining $r-1$ bars together with the n objects occupy $r-1+n$ intermediate positions. And we must choose $r-1$ of these for the bars and the remaining n for the objects. Hence our task can be accomplished in

$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

different ways.

Example 3.23 Seven people enter an elevator that will stop at five floors. In how many different ways can the people leave the elevator if we are interested only in the number that depart at each floor, and do not distinguish among the people? According to our general formula, the answer is

$$\binom{7+5-1}{7} = \binom{11}{7} = 330.$$

Suppose we are interested in finding the number of such possibilities in which at least one person gets off at each floor. We can then arbitrarily assign one person to get off at each floor, and the remaining two can get off at any floor. They can get off the elevator in

$$\binom{2+5-1}{2} = \binom{6}{2} = 15$$

different ways. ◇

Exercises

1. The survey discussed in Exercise 8 has been enlarged to include a fourth magazine D. It was found that no one who reads either magazine A or magazine B reads magazine D. However, 10 per cent of the people read magazine D and 5 per cent read both C and D. What per cent of the people do not read any magazine?

   [Ans. 5 per cent.]

2. A certain college administers three qualifying tests. They announce the following results: "Of the students taking the tests, 2 per cent failed all three tests, 6 per cent failed tests A and B, 5 per cent failed A and C, 8 per cent failed B and C, 29 per cent failed test A, 32 per cent failed B, and 16 per cent failed C." How many students passed all three qualifying tests?

3. Four partners in a game require a score of exactly 20 points to win. In how many ways can they accomplish this?

   [Ans. $\binom{23}{3}$.]

4. In how many ways can eight apples be distributed among four boys? In how many ways can this be done if each boy is to get at least one apple?

5. Suppose we have n balls and r boxes with $n \geq r$. Show that the number of different ways that the balls can be put into the boxes which insures that there is at least one ball in every box is $\binom{n-1}{r-1}$.

6. Identical prizes are to be distributed among five boys. It is observed that there are 15 ways that this can be done if each boy is to get at least one prize. How many prizes are there?

[Ans. 7.]

7. Let $p_1, p_2, \ldots, p_n$ be n statements relative to a possibility space $\mathcal{U}$. Show that the inclusion-exclusion formula gives a formula for the number of elements in the truth set of the disjunction $p_1 \vee p_2 \vee \ldots \vee p_n$ in terms of the numbers of elements in the truth sets of conjunctions formed from subsets of these statements.

8. A boss asks his or her secretary to put letters written to seven different persons into addressed envelopes. Find the number of ways that this can be done so that at least one person gets his or her own letter. [Hint: Use the result of Exercise 7, letting p_i be the statement "The ith person gets his or her own letter".]

[Ans. 3186.]

9. Consider the numbers from 2 to 10 inclusive. Let A_2 be the set of numbers divisible by 2 and A_3 the set of numbers divisible by 3. Find $n(A_2 \cup A_3)$ by using the inclusion-exclusion formula. From this result find the number of prime numbers between 2 and 10 (where a prime number is a number divisible only by itself and by 1). [Hint: Be sure to count the numbers 2 and 3 among the primes.]

10. Use the method of Exercise 9 to find the number of prime numbers between 2 and 100 inclusive. [Hint: Consider first the sets A_2, A_3, A_5, and A_7.]

[Ans. 25.]

Suggested reading.

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