Counting partitions

Up to now we have not had occasion to consider the partitions $[\{1,2\},\{3,4\}]$ and $[\{3,4\},\{1,2\}]$ of the integers from 1 to 4 as being different partitions. Here it will be convenient to do so, and to indicate this distinction we shall use the term *ordered partition*. An *ordered partition with r cells* is a partition with r cells (some of which may be empty), with a particular order specified for the cells.

We are interested in counting the number of possible ordered partitions with r cells that can be formed from a set of n objects having a prescribed number of elements in each cell. We consider first a special case to illustrate the general procedure.

Suppose that we have eight students, A, B, C, D, E, F, G, and H, and we wish to assign these to three rooms, Room 1, which is a triple room, Room 2, a triple room, and Room 3, a double room. In how many different ways can the assignment be made? One way to assign the students is to put them in the rooms in the order in which they arrive, putting the first three in Room 1, the next three in Room 2, and the last two in Room 3. There are 8! ways in which the students can arrive, but not all of these lead to different assignments. We can represent the assignment corresponding to a particular order of arrival as follows,

$$|BCA|DFE|HG|.$$

In this case, B, C, and A are assigned to Room 1, D, F, and E to Room 2, and H and G to Room 3. Notice that orders of arrival which simply change the order within the rooms lead to the same assignment. The number of different orders of arrival which lead to the same assignment as the one above is the number of arrangements which differ from the given one only in that the arrangement within the rooms is different. There are $3! \cdot 3! \cdot 2!$ such orders of arrival, since we can arrange the three in Room 1 in 3! different ways, for each of these the ones in Room 2 in 3! different ways, and for each of these, the ones in Room

3 in 2! ways. Thus we can divide the 8! different orders of arrival into groups of $3! \cdot 3! \cdot 2!$ different orders such that all the orders of arrival in a single group lead to the same room assignment. Since there are $3! \cdot 3! \cdot 2!$ elements in each group and 8! elements altogether, there are $\frac{8!}{3!3!2!}$ groups, or this many different room assignments.

The same argument could be carried out for n elements and r rooms, with n_1 in the first, n_2 in the second, etc. This would lead to the following result. Let $n_1, n_2, \ldots, n_r$ be nonnegative integers with

$$n_1 + n_2 + \ldots + n_r = n.$$

Then:

The number of ordered partitions with r cells $[A_1, A_2, \ldots, A_r]$ of a set of n elements with n_1 in the first cell, n_2 in the second, etc. is

$$\frac{n!}{n_1!n_2! \ldots n_r!}$$

We shall denote this number by the symbol

$$\binom{n!}{n_1!, n_2!, \ldots, n_r!}.$$

Note that this symbol is defined only if $n_1 + n_2 + \ldots + n_r = n$.

The special case of two cells is particularly important. Here the problem can be stated equivalently as the problem of finding the number of subsets with r elements that can be chosen from a set of n elements. This is true because any choice defines a partition $[A, \tilde{A}]$, where A is the set of elements chosen and $\tilde{A}$ is the set of remaining elements.

The number of such partitions is $\frac{n!}{r!(n-r)!}$ and hence this is also the number of subsets with r elements. Our notation $\binom{n}{r, n-r}$ for this case is shortened to $\binom{n}{r}$.

Notice that $\binom{n}{n-r}$ is the number of subsets with $n - r$ elements which can be chosen from n, which is the number of partitions of the form $[\tilde{A}, A]$ above. Clearly, this is the same as the number of $[A, \tilde{A}]$ partitions. Hence $\binom{n}{r} = \binom{n}{n-r}$.

Example 3.13 A college has scheduled six football games during a season. How many ways can the season end in two wins, three losses, and one tie? From each possible outcome of the season, we form a partition, with three cells, of the opposing teams. In the first cell we put the teams which our college defeats, in the second the teams to which our college loses, and in the third cell the teams which our college ties. There are $\binom{6}{2,3,1} = 60$ such partitions, and hence 60 ways in which the season can end with two wins, three losses, and one tie. $\diamondsuit$

Example 3.14 In the game of bridge, the hands N, E, S, and W determine a partition of the 52 cards having four cells, each with 13 elements. Thus there are $\frac{52!}{13!13!13!13!}$ different bridge deals. This number is about $5.3645 \cdot 10^{28}$, or approximately 54 billion billion billion deals. $\diamondsuit$

Example 3.15 The following example will be important in probability theory, which we take up in the next chapter. If a coin is thrown six times, there are 2^6 possibilities for the outcome of the six throws, since each throw can result in either a head or a tail. How many of these possibilities result in four heads and two tails? Each sequence of six heads and tails determines a two-cell partition of the numbers from one to six as follows: In the first cell put the numbers corresponding to throws which resulted in a head, and in the second put the numbers corresponding to throws which resulted in tails. We require that the first cell should contain four elements and the second two elements. Hence the number of the 2^6 possibilities which lead to four heads and two tails is the number of two-cell partitions of six elements which have four elements in the first cell and two in the second cell. The answer is $\binom{6}{4} = 15$. For n throws of a coin, a similar analysis shows that there are $\binom{n}{r}$ different sequences of H's and T's of length n which have exactly r heads and $n - r$ tails. $\diamondsuit$

Supplementary exercises.

Consider a town in which there are three plumbers, A, B, and C. On a certain day six residents of the town telephone for a plumber. If each resident selects a plumber from the telephone directory, in how many ways can it happen that

(a) Three residents call A, two residents call B, and one resident calls C?

[Ans. 60.]

(b) The distribution of calls to the plumbers is three, two, and one?

[Ans. 360.]

Some properties of the numbers $\binom{n}{j}$.

The numbers $\binom{n}{j}$ introduced in the previous section will play an important role in our future work. We give here some of the more important properties of these numbers.

A convenient way to obtain these numbers is given by the famous Pascal triangle, shown in Figure 3.5. To obtain the triangle we first write the 1's down the sides. Any of the other numbers in the triangle has the property that it is the sum of the two adjacent numbers in the row just above. Thus the next row in the triangle is 1, 6, 15, 20, 15, 6, 1. To find the number $\binom{n}{j}$ we look in the row corresponding to the number n and see where the diagonal line corresponding to the value of j intersects this row. For example, $\binom{4}{2} = 6$ is in the row marked $n = 4$ and on the diagonal marked $j = 2$.

The property of the numbers $\binom{n}{j}$ upon which the triangle is based is

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}.$$

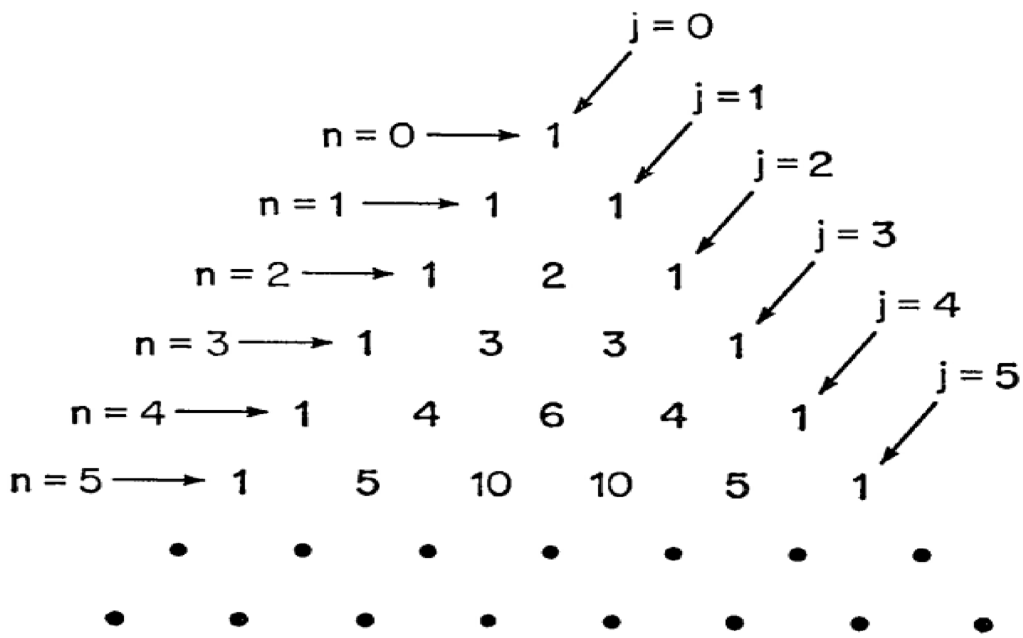

Figure 3.5: ◇

This fact can be verified directly (see Exercise 6), but the following argument is interesting in itself. The number $\binom{n+1}{j}$ is the number of subsets with j elements that can be formed from a set of $n+1$ elements. Select one of the $n+1$ elements, x. The $\binom{n+1}{j}$ subsets can be partitioned into those that contain x and those that do not. The latter are subsets of j elements formed from n objects, and hence there are $\binom{n}{j}$ such subsets. The former are constructed by adding x to a subset of $j-1$ elements formed from n elements, and hence there are $\binom{n}{j-1}$ of them. Thus

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}.$$

If we look again at the Pascal triangle, we observe that the numbers in a given row increase for a while, and then decrease. We can prove this fact in general by considering the ratio of two successive terms,

$$\frac{\binom{n}{j+1}}{\binom{n}{j}} = \frac{n!}{(j+1)!(n-j-1)!} \cdot \frac{j!(n-j)!}{n!} = \frac{n-j}{j+1}.$$

The numbers increase as long as the ratio is greater than 1, i.e., $n-j > j+1$. This means that $j < \frac{1}{2}(n-1)$. We must distinguish the case of an even n from an odd n. For example, if $n = 10$, j must be less than $\frac{1}{2}(10-1) = 4.5$. Hence for j up to 4 the terms are increasing, from $j = 5$ on, the terms decrease. For $n = 11$, j must be less than $\frac{1}{2}(11-1) = 5$. For $j = 5$, $(11-j)/(j+1) = 1$. Hence, up to $j = 5$ the terms increase, then $\binom{11}{5} = \binom{11}{6}$, and then the terms decrease.

Exercises

1. Extend the Pascal triangle to $n = 16$. Save the result for later use.

2. Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n,$$

   using the fact that a set with n elements has 2^n subsets.

3. For a set of ten elements prove that there are more subsets with five elements than there are subsets with any other fixed number of elements.

4. Using the fact that $\binom{n}{r+1} = \frac{n-r}{r+1} \cdot \binom{n}{r}$, compute $\binom{30}{s}$ for $s = 1, 2, 3, 4$ from the fact that $\binom{30}{0} = 1$.

   [Ans. 30; 435; 4060; 27,405.]

5. There are $\binom{52}{13}$ different possible bridge hands. Assume that a list is made showing all these hands, and that in this list the first card in every hand is crossed out. This leaves us with a list of twelve-card hands. Prove that at least two hands in the latter list contain exactly the same cards.

6. Prove that

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}$$

   using only the fact that

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

7. Construct a triangle in the same way that the Pascal triangle was constructed, except that whenever you add two numbers, use parity addition (table (a) in Figure 2.12). Construct the triangle for 16 rows. What does this triangle tell you about the numbers in the Pascal triangle? Use this result to check your triangle in Exercise 1.

8. In the triangle obtained in Exercise 7, what property do the rows 1, 2, 4, 8, and 16 have in common? What does this say about the numbers in the corresponding rows of the Pascal triangle? What would you predict for the terms in the 32nd row of the Pascal triangle?

9. For the following table state how one row is obtained from the preceding row and give the relation of this table to the Pascal triangle.

1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462
1	7	28	84	210	462	924

10. Referring to the table in Exercise 9, number the columns starting with $0, 1, 2, \ldots$ and number the rows starting with $1, 2, 3, \ldots$. Let $f(n, r)$ be the element in the nth column and the rth row. The table was constructed by the rule

$$f(n, r) = f(n-1, r) + f(n, r-1)$$

for $n > 0$ and $r > 1$, and $f(n, 1) = f(0, r) = 1$ for all n and r. Verify that

$$f(n, r) = \binom{n+r-1}{n}$$

satisfies these conditions and is in fact the only choice for $f(n, r)$ which will satisfy the conditions.

11. Consider a set $\{a_1, a_2, a_3\}$ of three objects which cannot be distinguished from one another. Then the ordered partitions with two cells which could be distinguished are: $[\{a_1, a_2, a_3\}, \emptyset]$, $[\{a_1, a_2\}, \{a_3\}]$, $[\{a_1\}, \{a_2, a_3\}]$, $[\emptyset, \{a_1, a_2, a_3\}]$. List all such ordered partitions with three cells. How many are there?

12. Let $f(n, r)$ be the number of distinguishable ordered partitions with r cells which can be formed from a set of n indistinguishable objects. Show that $f(n, r)$ satisfies the conditions

$$f(n, r) = f(n-1, r) + f(n, r-1)$$

for $n > 0$ and $r > 1$, and $f(n, 1) = f(0, r) = 1$ for all n and r. [Hint: Show that $f(n, r-1)$ is the number of partitions which have the last cell empty and $f(n-1, r)$ is the number which have at least one element in the last cell.]

13. Using the results of Exercises 10 and 12, show that the number of distinguishable ordered partitions with r cells which can be formed from a set of n indistinguishable objects is

$$\binom{n+r-1}{n}.$$

14. Assume that a mail carrier has seven letters to put in three mail boxes. How many ways can this be done if the letters are not distinguished?

[Ans. 36.]

15. For $n \geq r \geq k \geq s$ show that the identity

$$\binom{n}{r}\binom{r}{k}\binom{k}{s} = \binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{r-k}.$$

holds by replacing each binomial coefficient by a ratio of factorials.

16. Establish the identity in Exercise 15 in another way by showing that the two sides of the expression are simply two different ways of counting the number of solutions to the following problem: From a set of n people a subset of r is to be chosen; from the set of r people a subset of k is to be chosen; and from the set of k people a subset of s people is to be chosen.

17. Generalize the identity in Exercises 15 and 16 to solve the problem of finding the number of ways of selecting a t-element subset from an s-element subset from a k-element subset from an r-element subset of an n-element set, where $n \geq r \geq k \geq s \geq t$.

Binomial and multinomial theorems

It is sometimes necessary to expand products of the form $(x+y)^3$, $(x+2y+11z)^5$, etc. In this section we shall consider systematic ways of carrying out such expansions.

Consider first the special case $(x+y)^3$. We write this as

$$(x+y)^3 = (x+y)(x+y)(x+y).$$

To perform the multiplication, we choose either an x or a y from each of the three factors and multiply our choices together; we do this for all possible choices and add the results. We represent a particular set of choices by a two-cell partition of the numbers 1, 2, 3. In the first cell we put the numbers which correspond to factors from which we chose an x. In the second cell we put the numbers which correspond to factors from which we chose a y. For example, the partition $[\{1,3\},\{2\}]$ corresponds to a choice of x from the first and third factors and y from the second. The product so obtained is $xyx = x^2y$. The coefficient of x^2y in the expansion of $(x+y)^3$ will be the number of partitions which lead to a choice of two x's and one y, that is, the number of two-cell partitions of three elements with two elements in the first cell and one in the second, which is $\binom{3}{2} = 3$. More generally, the coefficient of the term of the form $x^j y^{3-j}$ will be $\binom{3}{j}$ for $j = 0, 1, 2, 3$. Thus we can write the desired expansion as

$$\begin{aligned}(x+y)^3 &= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

The same argument carried out for the expansion $(x+y)^n$ leads to the binomial theorem of algebra.

Binomial theorem. The expansion of $(x+y)^n$ is given by

$$\binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \ldots + \binom{n}{1}xy^{n-1} + \binom{n}{0}y^n.$$

Example 3.16 Let us find the expansion for $(a-2b)^3$. To fit this into the binomial theorem, we think of x as being a and y as being $-2b$. Then we have

$$\begin{aligned}(a-2b)^3 &= a^3 + 3a^2(-2b) + 3a(-2b)^2 + (-2b)^3 \\ &= a^3 - 6a^2b + 12ab^2 - 8b^3.\end{aligned}$$

We turn now to the problem of expanding the trinomial $(x+y+z)^3$. Again we write

$$(x+y+z)^3 = (x+y+z)(x+y+z)(x+y+z).$$

This time we choose either an x or y or z from each of the three factors. Our choice is now represented by a three-cell partition of the set of numbers $\{1,2,3\}$. The first cell has the numbers corresponding to factors from which we choose an x, the second cell the numbers corresponding to factors from which we choose a y, and the third those from which we choose a z. For example, the partition $[\{1,3\},\emptyset,\{2\}]$ corresponds to a choice of x from the first and third factors, no y's, and a z from the second factor. The term obtained is $xzx = x^2z$. The coefficient of the term x^2z in the expansion is thus the number of three-cell partitions with two elements in the first cell, none in the second, and one in the third. There are $\binom{3}{2,0,1} = 3$ such partitions. In general, the coefficient of the term of the form $x^a y^b z^c$ in the expansion of $(x+y+z)^3$ will be

$$\binom{3}{a,b,c} = \frac{3!}{a!b!c!}.$$

Finding this way the coefficient for each possible a, b, and c, we obtain

$$(x+y+z)^3 = x^3+y^3+x^3+3x^2y+3xy^2+3yz^2+3y^2z+3xz^2+3x^2z+6xyz.$$

The same method can be carried out in general for finding the expansion of $(x_1+x_2+\ldots+x_r)^n$. From each factor we choose either an x_1, or x_2, $\ldots$, or x_r, form the product and add these products for all possible choices. We will have r^n products, but many will be equal. A particular choice of one term from each factor determines an r-cell partition of the numbers from 1 to n. In the first cell we put the numbers of the factors from which we choose an x_1, in the second cell those from which we choose x_2, etc. A particular choice gives us a term of the form $x_1^{n_1}x_2^{n_2}\ldots x_r^{n_r}$ with $n_1+n_2+\ldots+n_r = n$. The corresponding partition has n_1 elements in the first cell, n_2 in the second, etc. For each such partition we obtain one such term. Hence the number of these terms which we obtain is the number of such partitions, which is

$$\binom{n}{n_1,n_2,\ldots,n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}.$$

Thus we have the multinomial theorem.

Multinomial theorem. The expansion of $(x_1 + x_2 + \ldots + x_r)^n$ is found by adding all terms of the form

$$\binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

where $n_1 + n_2 + \ldots + n_r = n$.

Exercises

1. Expand by the binomial theorem
   (a) $(x + y)^4$.
   (b) $(1 + x)^5$.
   (c) $(x - y)^3$.
   (d) $(2x + a)^4$.
   (e) $(2x - 3y)^3$.
   (f) $(100 - 1)^5$.

2. Expand
   (a) $(x + y + x)^4$.
   (b) $(2x + y - z)^3$.
   (c) $(2 + 2 + 1)^3$. (Evaluate two ways.)

3. (a) Find the coefficient of the term $x^2y^3z^2$ in the expansion of $(x + y + z)^7$.

   [Ans. 210.]

   (b) Find the coefficient of the term $x^6y^3z^2$ in the expression $(x - 2y + 5z)^{11}$

   [Ans. -924,000.]

4. Using the binomial theorem prove that
   (a)

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n.$$

(b)

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots \pm \binom{n}{n} = 0$$

for $n > 0$.

5. Using an argument similar to the one in Section 3.6, prove that

$$\binom{n+1}{i,j,k} = \binom{n}{i-1,j,k} + \binom{n}{i,j-1,k} + \binom{n}{i,j,k-1}.$$

6. Let $f(n, r)$ be the number of terms in the multinomial expansion of

$$(x_1 + x_2 + \ldots + x_r)^n$$

and show that

$$f(n, r) = \binom{n+r-1}{n}.$$

[Hint: Show that the conditions of Exercise 10 are satisfied by showing that $f(n, r-1)$ is the number of terms which do not have x_r and $f(n-1, r)$ is the number which do.]

7. How many terms are there in each of the expansions:

(a) $(x + y + z)^6$?

[Ans. 28.]

(b) $(a + 2b + 5c + d)^4$?

[Ans. 35.]

(c) $(r + s + t + u + v)^6$?

[Ans. 210.]

8. Prove that k^n is the sum of the numbers $\binom{n}{r_1,r_2,\ldots,r_k}$ for all choices of $r_1, r_2, \ldots, r_k$ such that

$$r_1 + r_2 + \ldots + r_k = n.$$

Supplementary exercises.

If $a+b+c=n$, show that

$$\binom{n}{a,b,c}=\binom{n}{a}\binom{n-a}{b}.$$

If $a+b+c+d=n$, show that

$$\binom{n}{a,b,c,d}=\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}.$$