Two-digit number systems

In the decimal number system one can write any number by using only the ten digits, $0, 1, 2, \ldots, 9$. Other number systems can be constructed which use either fewer or more digits. Probably the simplest number system is the binary number system which uses only the digits 0 and 1. We shall consider all the possible ways of forming number systems using only these two digits.

The two basic arithmetical operations are addition and multiplication. To understand any arithmetic system, it is necessary to know how to add or multiply any two digits together. Thus to understand the decimal system, we had to learn a multiplication table and an addition table, each of which had 100 entries. To understand the binary system, we have to learn a multiplication and an addition table, each of which has only four entries. These are shown in Figure 2.11. The multiplication table given there is completely determined by the two familiar rules that multiplying a number by zero gives zero, and multiplying a number by one leaves it unchanged. For addition, we have only the rule that the addition of zero to a number does not change that number. The latter rule is sufficient to determine all but one of the entries in the addition table in Figure 2.11. We must still decide what shall be the sum $1 + 1$.

What are the possible ways in which we can complete the addition table? The only one-digit numbers that we can use are 0 and l, and these lead to interesting systems. Of the possible two-digit numbers, we see that 00 and 01 are the same as 0 and l and so do not give anything new. The number 11 or any greater number would introduce a “jump” in the table, hence the only other possibility is 10. The addition tables of these three different number systems are shown in Figure 2.12, and they all have the multiplication table shown in Figure 2.11. Each of these systems is interesting in itself as the interpretations below show.

Let us say that the *parity* of a positive integer is the fact of its being

+	0	1
0	0	1
1	1	0

(a)

+	0	1
0	0	1
1	1	1

(b)

+	0	1
0	0	1
1	1	10

(c)

Figure 2.12: ◇

odd or even. Consider now the number system having the addition table (a) in Figure 2.12 and let 0 represent "even" and 1 represent "odd". The tables above now tell how the parity of a combination of two positive integers is related to the parity of each. Thus $0 \cdot 1 = 0$ tells us that the product of an even number and an odd number is even, while $1 + 1 = 0$ tells us that the sum of two odd numbers is even, etc. Thus the first number system is that which we get from the arithmetic of the positive integers if we consider only the parity of numbers.

The second number system, which has the addition table (b) in Figure 2.12, has an interpretation in terms of sets. Let 0 correspond to the empty set $\emptyset$ and 1 correspond to the universal set $\mathcal{U}$. Let the addition of numbers correspond to the union of sets and let the multiplication of sets correspond to the intersection of sets. Then $0 \cdot 1 = 0$ tells us that $\emptyset \cap \mathcal{U} = \emptyset$ and $1 + 1 = 1$ tells us that $\mathcal{U} \cup \mathcal{U} = \mathcal{U}$. The student should give the interpretations for the other arithmetic computations possible for this number system.

Finally, the third number system, which has the addition table in (c) of Figure 2.12, is the so-called *binary number system*. Every ordinary integer can be written as a binary integer. Thus the binary 0 corresponds to the ordinary 0, and the binary unit 1 to the ordinary single unit. The binary number 10 means a "unit of higher order" and corresponds to the ordinary number two (not to ten). The binary number 100 then means two times two or four. In general, if $b_n b_{n-1} \ldots b_2 b_1 b_0$ is a binary number, where each digit is either 0 or 1, then the corresponding ordinary integer I is given by the formula

$$I = b_n \cdot 2^n + b_{n-1} \cdot 2^{n-1} + \ldots + b_2 \cdot 2^2 + b_1 \cdot 2 + b_0.$$

Thus the binary number 11001 corresponds to $2^4 + 2^3 + 1 = 16 + 8 + 1 = 25$. The table in Figure 2.13 shows some binary numbers and their decimal equivalents.

Because electronic circuits are particularly well adapted to performing computations in the binary system, modern high-speed electronic computers are frequently constructed to work in the binary system.

Example 2.6 As an example of a computation, let us multiply 5 by 5 in the binary system. Since the binary equivalent of 5 is the number 101, the multiplication is done as follows.

$$\begin{array}{ccccc} & & 1 & 0 & 1 \\ & & 1 & 0 & 1 \\ \hline & & 1 & 0 & 1 \\ & 0 & 0 & 0 & \\ 1 & 0 & 1 & & \\ \hline 1 & 1 & 0 & 0 & 1 \end{array}$$

The answer is the binary number 11001, which we saw above was equivalent to the decimal integer 25, the answer we expected to get. ♢

Voting coalitions

As an application of our set concepts, we shall consider the significance of voting coalitions in voting bodies. Here the universal set is a set of human beings which form a decision-making body. For example, the universal set might be the members of a committee, or of a city council, or of a convention, or of the House of Representatives, etc. Each member can cast a certain number of votes. The decision as to whether or not a measure is passed can be decided by a simple majority rule, or two-thirds majority, etc.

Suppose now that a subset of the members of the body forms a coalition in order to pass a measure. The question is whether or not they have enough votes to guarantee passage of the measure. If they have enough votes to carry the measure, then we say they form a winning coalition. If the members not in the coalition can pass a measure of their own, then we say that the original coalition is a losing coalition. Finally, if the members of the coalition cannot carry their measure, and if the members not in the coalition cannot carry their measure, then the coalition is called a blocking coalition.

Let us restate these definitions in set-theoretic terms. A coalition C is winning if they have enough votes to carry an issue; coalition C is losing if the coalition $\tilde{C}$ is winning; and coalition C is blocking if neither C nor $\tilde{C}$ is a winning coalition.

The following facts are immediate consequences of these definitions. The complement of a winning coalition is a losing coalition. The complement of a losing coalition is a winning coalition. The complement of a blocking coalition is a blocking coalition.

Example 2.7 A committee consists of six members each having one vote. A simple majority vote will carry an issue. Then any coalition of four or more members is winning, any coalition with one or two members is losing, and any three-person coalition is blocking. ◇

Example 2.8 Suppose in Example 2.7 one of the six members (say the chair) is given the additional power to break ties. Then any three-person coalition of which the chair is a member is winning, while the other three-person coalitions are losing; hence there are no blocking coalitions. The other coalitions are as in Example 2.7. ◇

Example 2.9 Let the universal set $\mathcal{U}$ be the set $\{x, y, w, z\}$, where x and y each has one vote, w has two votes, and z has three votes. Suppose it takes five votes to carry a measure. Then the winning coalitions are: $\{z, w\}$, $\{z, x, y\}$, $\{z, w, x\}$, $\{z, w, y\}$, and $\mathcal{U}$. The losing coalitions are the complements of these sets. Blocking coalitions are: $\{z\}$, $\{z, x\}$, $\{z, y\}$, $\{w, x\}$, $\{w, y\}$, and $\{w, x, y\}$. ◇

The last example shows that it is not always necessary to list all members of a winning coalition. For example, if the coalition $\{z, w\}$ is winning, then it is obvious that the coalition $\{z, w, y\}$ is also winning. In general, if a coalition C is winning, then any other set that has C as a subset will also be winning. Thus we are led to the notion of a minimal winning coalition. A minimal winning coalition is a winning coalition which contains no smaller winning coalition as a subset. Another way of stating this is that a minimal winning coalition is a winning coalition such that, if any member is lost from the coalition, then it ceases to be a winning coalition.

If we know the minimal winning coalitions, then we know everything that we need to know about the voting problem. The winning coalitions are all those sets that contain a minimal winning coalition, and the losing coalitions are the complements of the winning coalitions. All other sets are blocking coalitions.

In Example 2.7 the minimal winning coalitions are the sets containing four members. In Example 2.8 the minimal winning coalitions are the three-member coalitions that contain the tie-breaking member and the four-member coalitions that do not contain the tie-breaking member. The minimal winning coalitions in the third example are the sets $\{z, w\}$ and $\{z, x, y\}$.

Sometimes there are committee members who have special powers or lack of power. If a member can pass any measure he or she wishes without needing anyone else to vote with him or her, then we call him or her a dictator. Thus member x is a dictator if and only if $\{x\}$ is a winning coalition. A somewhat weaker but still very powerful member is one who can by himself or herself block any measure. If x is such a member, then we say that x has veto power. Thus x has veto power if and only if $\{x\}$ is a blocking coalition. Finally if x is not a member of any minimal winning coalition, we shall call him or her a powerless member. Thus x is powerless if and only if any winning coalition of which x is a member is a winning coalition without him or her.

Example 2.10 An interesting example of a decision-making body is the Security Council of the United Nations. (We discuss the rules prior to 1966.) The Security Council has eleven members consisting of the five permanent large-nation members called the Big Five, and six small-nation members. In order that a measure be passed by the Council, seven members including all of the Big Five must vote for the measure. Thus the seven-member sets made up of the Big Five plus two small nations are the minimal winning coalitions. Then the losing coalitions are the sets that contain at most four small nations. The blocking coalitions are the sets that are neither winning nor losing. In particular, a unit set that contains one of the Big Five as a member is a blocking coalition. This is the sense in which a Big Five member has a veto. [The possibility of “abstaining” is immaterial in the above discussion.]

In 1966 the number of small-nation members was increased to 10.

A measure now requires the vote of nine members, including all of the Big Five. (See Exercise 11.) ◇

Exercises

1. A committee has w, x, y, and z as members. Member w has two votes, the others have one vote each. List the winning, losing, and blocking coalitions.

2. A committee has n members, each with one vote. It takes a majority vote to carry an issue. What are the winning, losing, and blocking coalitions?

3. Rhe Board of Estimate of New York City consists (that is, consisted at one time) of eight members with voting strength as follows:

s.	Mayor	4 votes
t.	Controller	4
u.	Council President	4
v.	Brooklyn Borough President	2
w.	Manhattan Borough President	2
x.	Bronx Borough President	2
y.	Richmond Borough President	2
z.	Queens Borough President	2

   A simple majority is needed to carry an issue. List the minimal winning coalitions. List the blocking coalitions. Do the same if we give the mayor the additional power to break ties.

4. A company has issued 100,000 shares of common stock and each share has one vote. How many shares must a stockholder have to be a dictator? How many to have a veto?

   [Ans. 50,001; 50,000.]

5. In Exercise 4, if the company requires a two-thirds majority vote to carry an issue, how many shares must a stockholder have to be a dictator or to have a veto?

   [Ans. At least 66,667; at least 33,334.]

6. Prove that if a committee has a dictator as a member, then the remaining members are powerless.

7. We can define a maximal losing coalition in analogy to the minimal winning coalitions. What is the relation between the maximal losing and minimal winning coalitions? Do the maximal losing coalitions provide all relevant information?

8. Prove that any two minimal winning coalitions have at least one member in common.

9. Find all the blocking coalitions in the Security Council example (Example 2.10).

10. Prove that if a member has veto power and if he or she together with any one other member can carry a measure, then the distribution of the remaining votes is irrelevant.

11. Find the winning, losing, and blocking coalitions in the Security Council, using the revised (1966) structure.

Suggested reading.

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