Sets and subsets

Introduction

A well-defined collection of objects is known as a *set*. This concept, in its complete generality, is of great importance in mathematics since all of mathematics can be developed by starting from it.

The various pieces of furniture in a given room form a set. So do the books in a given library, or the integers between 1 and 1,000,000, or all the ideas that mankind has had, or the human beings alive between one billion B.C. and ten billion A.D. These examples are all examples of finite sets, that is, sets having a finite number of elements. All the sets discussed in this book will be finite sets.

There are two essentially different ways of specifying a set. One can give a rule by which it can be determined whether or not a given object is a member of the set, or one can give a complete list of the elements in the set. We shall say that the former is a *description* of the set and the latter is a *listing* of the set. For example, we can define a set of four people as (a) the members of the string quartet which played in town last night, or (b) four particular persons whose names are Jones, Smith, Brown, and Green. It is customary to use braces to, surround the listing of a set; thus the set above should be listed $\{\text{Jones}, \text{Smith}, \text{Brown}, \text{Green}\}$.

We shall frequently be interested in sets of logical possibilities, since the analysis of such sets is very often a major task in the solving of a problem. Suppose, for example, that we were interested in the successes of three candidates who enter the presidential primaries (we assume there are no other entries). Suppose that the key primaries will be held in New Hampshire, Minnesota, Wisconsin, and California. Assume that candidate A enters all the primaries, that B does not contest in New Hampshire's primary, and C does not contest in Wisconsin's. A list of the logical possibilities is given in Figure 2.1. Since the New Hampshire and Wisconsin primaries can each end in two ways, and the Minnesota and California primaries can each end in three ways, there are in all $2 \cdot 2 \cdot 3 \cdot 3 = 36$ different logical possibilities as listed in Figure 2.1.

A set that consists of some members of another set is called a *subset* of that set. For example, the set of those logical possibilities in Figure 2.1 for which the statement "Candidate A wins at least three primaries" is true, is a subset of the set of all logical possibilities. This subset can also be defined by listing its members: $\{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}, \mathrm{P4}, \mathrm{P7}, \mathrm{P13}, \mathrm{P19}\}$.

In order to discuss all the subsets of a given set, let us introduce the following terminology. We shall call the original set the *universal set*, one-element subsets will be called *unit* sets, and the set which contains no members the *empty set*. We do not introduce special names for other kinds of subsets of the universal set. As an example, let the universal set $\mathcal{U}$ consist of the three elements $\{a, b, c\}$. The proper subsets of $\mathcal{U}$ are those sets containing some but not all of the elements of $\mathcal{U}$. The proper subsets consist of three two-element sets namely, $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$ and three unit sets, namely, $\{a\}$, $\{b\}$, and $\{c\}$. To complete the picture, we also consider the universal set a subset (but not a proper subset) of itself, and we consider the empty set $\emptyset$, that contains no elements of $\mathcal{U}$, as a subset of $\mathcal{U}$. At first it may seem strange that we should include the sets $\mathcal{U}$ and $\emptyset$ as subsets of $\mathcal{U}$, but the reasons for their inclusion will become clear later.

We saw that the three-element set above had $8 = 2^3$ subsets. In general, a set with n elements has 2^n subsets, as can be seen in the following manner. We form subsets P of $\mathcal{U}$ by considering each of the elements of $\mathcal{U}$ in turn and deciding whether or not to include it in the subset P. If we decide to put every element of $\mathcal{U}$ into P, we get the universal set, and if we decide to put no element of $\mathcal{U}$ into P, we get the empty set. In most cases we will put some but not all the elements into P and thus obtain a proper subset of $\mathcal{U}$. We have to make n decisions, one for each element of the set, and for each decision we have to choose between two alternatives. We can make these decisions in $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ ways, and hence this is the number of different subsets of $\mathcal{U}$ that can be formed. Observe that our formula would not have been so simple if we had not included the universal set and the empty set as subsets of $\mathcal{U}$.

Possibility Number	Winner in New Hampshire	Winner in Minnesota	Winner in Wisconsin	Winner in California
P1	A	A	A	A
P2	A	A	A	B
P3	A	A	A	C
P4	A	A	B	A
P5	A	A	B	B
P6	A	A	B	C
P7	A	B	A	A
P8	A	B	A	B
P9	A	B	A	C
P10	A	B	B	A
P11	A	B	B	B
P12	A	B	B	C
P13	A	C	A	A
P14	A	C	A	B
P15	A	C	A	C
P16	A	C	B	A
P17	A	C	B	B
P18	A	C	B	C
P19	C	A	A	A
P20	C	A	A	B
P21	C	A	A	C
P22	C	A	B	A
P23	C	A	B	B
P24	C	A	B	C
P25	C	B	A	A
P26	C	B	A	B
P27	C	B	A	C
P28	C	B	B	A
P29	C	B	B	B
P30	C	B	B	C
P31	C	C	A	A
P32	C	C	A	B
P33	C	C	A	C
P34	C	C	B	A
P35	C	C	B	B
P36	C	C	B	C

Figure 2.1: ◇

In the example of the voting primaries above there are 2^{36} or about 70 billion subsets. Of course, we cannot deal with this many subsets in a practical problem, but fortunately we are usually interested in only a few of the subsets. The most interesting subsets are those which can be defined by means of a simple rule such as "the set of all logical possibilities in which C loses at least two primaries". It would be difficult to give a simple description for the subset containing the elements $\{\mathrm{P1}, \mathrm{P4}, \mathrm{P14}, \mathrm{P30}, \mathrm{P34}\}$. On the other hand, we shall see in the next section how to define new subsets in terms of subsets already defined.

Example 2.1 We illustrate the two different ways of specifying sets in terms of the primary voting example. Let the universal set $\mathcal{U}$ be the logical possibilities given in Figure 2.1.

1. What is the subset of $\mathcal{U}$ in which candidate B wins more primaries than either of the other candidates?

   [Ans. $\{\mathrm{P11}, \mathrm{P12}, \mathrm{P17}, \mathrm{P23}, \mathrm{P26}, \mathrm{P28}, \mathrm{P29}\}$.]

2. What is the subset in which the primaries are split two and two?

   [Ans. $\{\mathrm{P5}, \mathrm{P8}, \mathrm{P10}, \mathrm{P15}, \mathrm{P21}, \mathrm{P30}, \mathrm{P31}, \mathrm{P35}\}$.]

3. Describe the set $\{\mathrm{P1}, \mathrm{P4}, \mathrm{P19}, \mathrm{P22}\}$.

   [Ans. The set of possibilities for which A wins in Minnesota and California.]

4. How can we describe the set $\{\mathrm{P18}, \mathrm{P24}, \mathrm{P27}\}$

   [Ans. The set of possibilities for which C wins in California, and the other primaries are split three ways.]

$\diamond$

Exercises

1. In the primary example, give a listing for each of the following sets.

   (a) The set in which C wins at least two primaries.

(b) The set in which the first three primaries are won by the same candidate.

(c) The set in which B wins all four primaries.

2. The primaries are considered decisive if a candidate can win three primaries, or if he or she wins two primaries including California. List the set in which the primaries are decisive.

3. Give simple descriptions for the following sets (referring to the primary example).

   (a) $\{P33, P36\}$.

   (b) $\{P10, P11, P12, P28, P29, P30\}$.

   (c) $\{P6, P20, P22\}$.

4. Joe, Jim, Pete, Mary, and Peg are to be photographed. They want to line up so that boys and girls alternate. List the set of all possibilities.

5. In Exercise 4, list the following subsets.

   (a) The set in which Pete and Mary are next to each other.

   (b) The set in which Peg is between Joe and Jim.

   (c) The set in which Jim is in the middle.

   (d) The set in which Mary is in the middle.

   (e) The set in which a boy is at each end.

6. Pick out all pairs in Exercise 5 in which one set is a subset of the other.

7. A TV producer is planning a half-hour show. He or she wants to have a combination of comedy, music, and commercials. If each is allotted a multiple of five minutes, construct the set of possible distributions of time. (Consider only the total time allotted to each.)

8. In Exercise 7, list the following subsets.

   (a) The set in which more time is devoted to comedy than to music.

(b) The set in which no more time is devoted to commercials than to either music or comedy.

(c) The set in which exactly five minutes is devoted to music.

(d) The set in which all three of the above conditions are satisfied.

9. In Exercise 8, find two sets, each of which is a proper subset of the set in 8a and also of the set in 8c.

10. Let $\mathcal{U}$ be the set of paths in Figure ??. Find the subset in which

    (a) Two balls of the same color are drawn.

    (b) Two different color balls are drawn.

11. A set has 101 elements. How many subsets does it have? How many of the subsets have an odd number of elements?

    [Ans. $2^{101}; 2^{100}$.]

12. Do Exercise 11 for the case of a set with 102 elements.

Operations on subsets

we considered the ways in which one could form new statements from given statements. Now we shall consider an analogous procedure, the formation of new sets from given sets. We shall assume that each of the sets that we use in the combination is a subset of some universal set, and we shall also want the newly formed set to be a subset of the same universal set. As usual, we can specify a newly formed set either by a description or by a listing.

If P and Q are two sets, we shall define a new set $P \cap Q$, called the *intersection* of P and Q, as follows: $P \cap Q$ is the set which contains those and only those elements which belong to both P and Q. As an example, consider the logical possibilities listed in Figure 2.1. Let P be the subset in which candidate A wins at least three primaries, i.e., the set $\{P1, P2, P3, P4, P7, P13, P19\}$; let Q be the subset in which A wins the first two primaries, i.e., the set $\{P1, P2, P3, P4, P5, P6\}$. Then the intersection $P \cap Q$ is the set in which both events take place, i.e., where A wins the first two primaries and wins at least three primaries. Thus $P \cap Q$ is the set $\{P1, P2, P3, P4\}$.

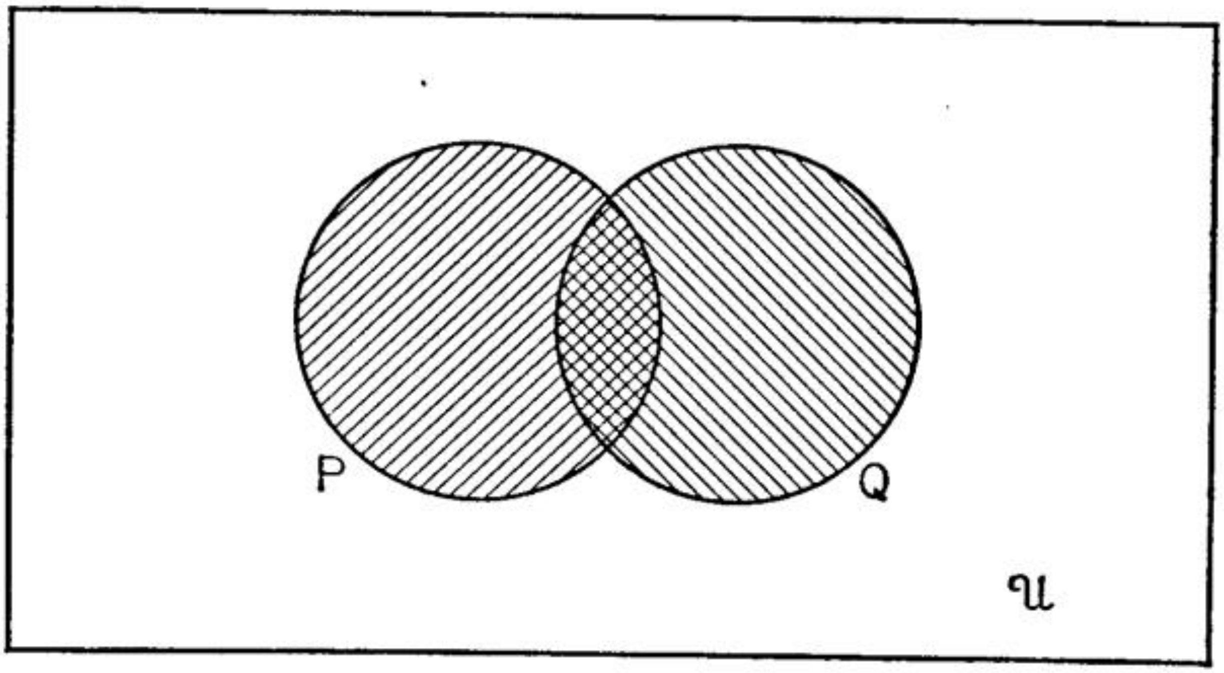

Figure 2.2: $\diamondsuit$

If P and Q are two sets, we shall define a new set $P \cup Q$ called the *union* of P and Q as follows: $P \cup Q$ is the set that contains those and only those elements that belong either to P or to Q (or to both). In the example in the paragraph above, the union $P \cup Q$ is the set of possibilities for which either A wins the first two primaries or wins at least three primaries, i.e., the set $\{P1, P2, P3, P4, P5, P6, P7, P13, P19\}$.

To help in visualizing these operations we shall draw diagrams, called *Venn diagrams*, which illustrate them. We let the universal set be a rectangle and let subsets be circles drawn inside the rectangle. In Figure 2.2 we show two sets P and Q as shaded circles. Then the doubly crosshatched area is the intersection $P \cap Q$ and the total shaded area is the union $P \cup Q$.

If P is a given subset of the universal set $\mathcal{U}$, we can define a new set $\tilde{P}$ called the *complement* of P as follows: P is the set of all elements of $\mathcal{U}$ that are not contained in P. For example, if, as above, Q is the set in which candidate A wins the first two primaries, then $\tilde{Q}$ is the set $\{\mathrm{P7}, \mathrm{P8}, \ldots, \mathrm{P36}\}$. The shaded area in Figure 2.3 is the complement of the set P. Observe that the complement of the empty set $\emptyset$ is the universal set $\mathcal{U}$, and also that the complement of the universal set is the empty set.

Sometimes we shall be interested in only part of the complement of a set. For example, we might wish to consider the part of the complement of the set Q that is contained in P, i.e., the set $P \cap \tilde{Q}$. The shaded area in Figure 2.4 is $P \cap \tilde{Q}$.

A somewhat more suggestive definition of this set can be given as

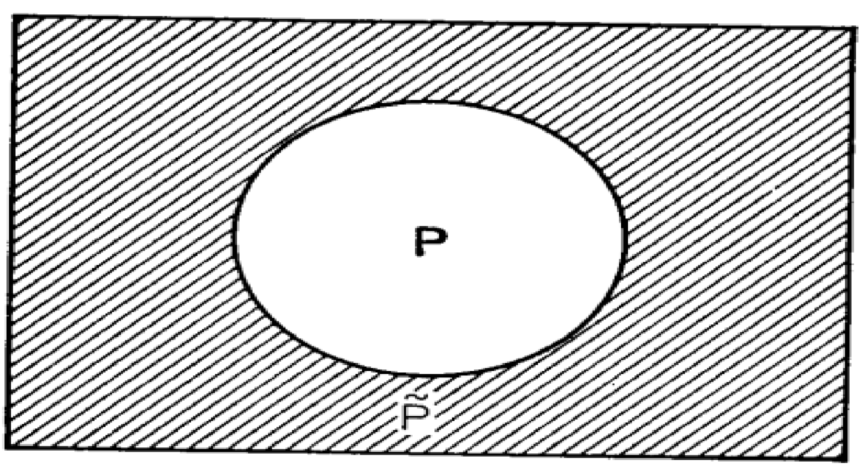

Figure 2.3: $\diamond$

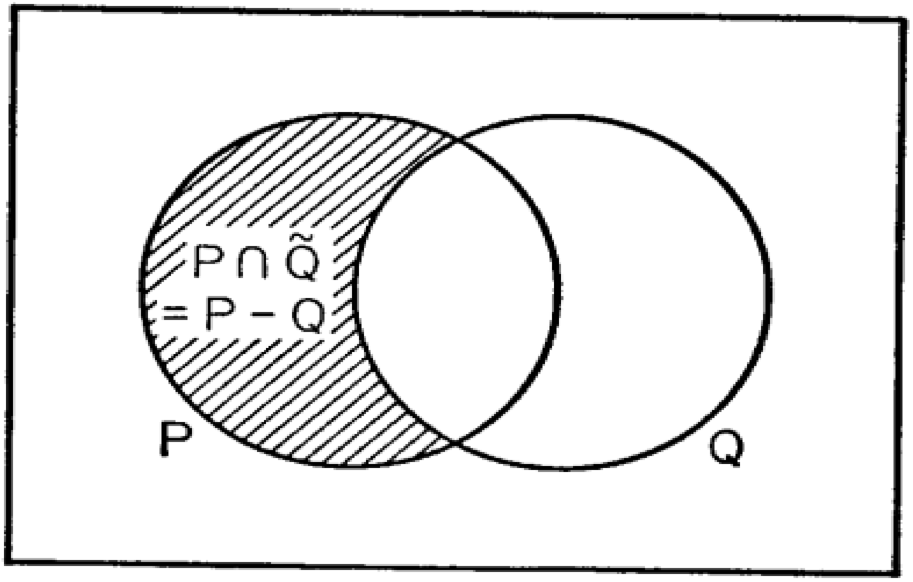

Figure 2.4: ◇

follows: Let $P-Q$ be the *difference* of P and Q, that is, the set that contains those elements of P that do not belong to Q. Figure 2.4 shows that $P \cap \tilde{Q}$ and $P-Q$ are the same set. In the primary voting example above, the set $P-Q$ can be listed as $\{\mathrm{P7}, \mathrm{P13}, \mathrm{P19}\}$.

The complement of a subset is a special case of a difference set, since we can write $\tilde{Q} = \mathcal{U} - Q$. If P and Q are nonempty subsets whose intersection is the empty set, i.e., $P \cap Q = \emptyset$, then we say that they are *disjoint* subsets.

Example 2.2 In the primary voting example let R be the set in which A wins the first three primaries, i e., the set $\{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\}$; let S be the set in which A wins the last two primaries, i.e., the set $\{\mathrm{P1}, \mathrm{P7}, \mathrm{P13}, \mathrm{P19}, \mathrm{P25}, \mathrm{P31}\}$. Then $R \cap S = \{\mathrm{P1}\}$ is the set in which A wins the first three primaries and also the last two, that is, he or she wins all the primaries. We also have

$$R \cup S = \{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}, \mathrm{P7}, \mathrm{P13}, \mathrm{P19}, \mathrm{P25}, \mathrm{P31}\},$$

which can be described as the set in which A wins the first three primaries or the last two. The set in which A does not win the first three primaries is $\tilde{R} = \{\text{P4}, \text{P5}, \ldots, \text{P36}\}$. Finally, we see that the difference set $R - S$ is the set in which A wins the first three primaries but not both of the last two. This set can be found by taking from R the element P1 which it has in common with S, so that $R - S = \{\text{P2}, \text{P3}\}$. $\diamondsuit$

The relationship between sets and compound statements

The reader may have observed several times in the preceding sections that there was a close connection between sets and statements, and between set operations and compounding operations. In this section we shall formalize these relationships.

If we have a number of statements relative to a set of logical possibilities, there is a natural way of assigning a set to each statement. First of all, we take the set of logical possibilities as our universal set. Then to each statement we assign the subset of logical possibilities of the universal set for which that statement is true. This idea is so important that we embody it in a formal definition.

Definition. Let $\mathcal{U}$ be a set of logical possibilities, let p be a statement relative to it, and let P be that subset of the possibilities for which p is true; then we call P the *truth set* of p.

If p and q are statements, then $p \vee q$ and $p \wedge q$ are also statements and hence must have truth sets. To find the truth set of $p \vee q$, we observe that it is true whenever p is true or q is true (or both). Therefore we must assign to $p \vee q$ the logical possibilities which are in P or in Q (or both); that is, we must assign to $p \vee q$ the set $P \cup Q$. On the other hand, the statement $p \wedge q$ is true only when both p and q are true, so that we must assign to $p \wedge q$ the set $P \cap Q$.

Thus we see that there is a close connection between the logical operation of disjunction and the set operation of union, and also between conjunction and intersection. A careful examination of the definitions of union and intersection shows that the word "or" occurs in the definition of union and the word "and" occurs in the definition of intersection. Thus the connection between the two theories is not surprising.

Since the connective "not" occurs in the definition of the complement of a set, it is not surprising that the truth set of $\neg p$ is $\tilde{P}$. This

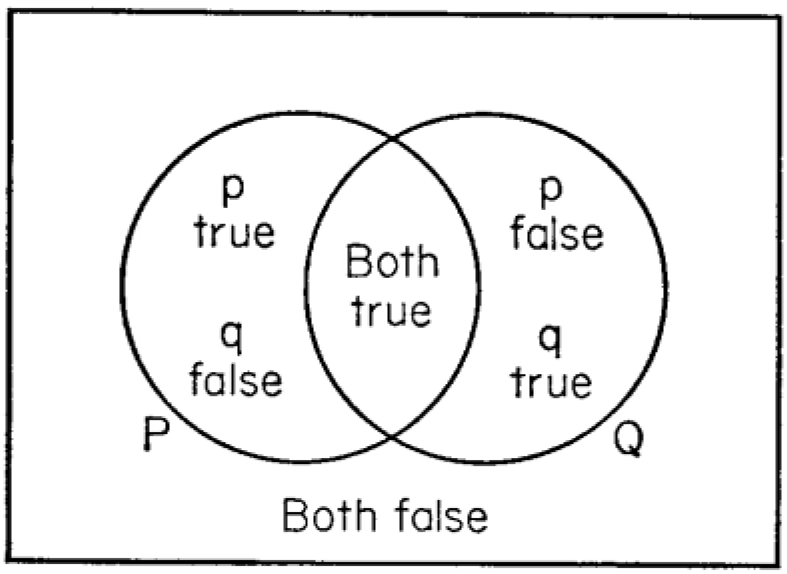

Figure 2.6: $\diamondsuit$

follows since $\neg p$ is true when p is false, so that the truth set of $\neg p$ contains all logical possibilities for which p is false, that is, the truth set of $\neg p$ is $\tilde{P}$.

The truth sets of two propositions p and q are shown in Figure 2.6. Also marked on the diagram are the various logical possibilities for these two statements. The reader should pick out in this diagram the truth sets of the statements $p \vee q$, $p \wedge q$, $\neg p$, and $\neg q$.

The connection between a statement and its truth set makes it possible to "translate" a problem about compound statements into a problem about sets. It is also possible to go in the reverse direction. Given a problem about sets, think of the universal set as being a set of logical possibilities and think of a subset as being the truth set of a statement. Hence we can "translate" a problem about sets into a problem about compound statements.

So far we have discussed only the truth sets assigned to compound statements involving $\vee$, $\wedge$, and $\neg$. All the other connectives can be defined in terms of these three basic ones, so that we can deduce what truth sets should be assigned to them. For example, we know that $p \to q$ is equivalent to $\neg p \vee q$ (see Figure **??**). Hence the truth set of $p \to q$ is the same as the truth set of $\neg p \vee q$, that is, it is $\tilde{P} \cup Q$. The Venn diagram for $p \to q$ is shown in Figure 2.7, where the shaded area is the truth set for the statement. Observe that the unshaded area in Figure 2.7 is the set $P - Q = P \cap \tilde{Q}$, which is the truth set of the statement $p \wedge \neg q$. Thus the shaded area is the set $\widetilde{P - Q} = \widetilde{P \cap \tilde{Q}}$, which is the truth set of the statement $\neg(p \wedge \neg q)$. We have thus discovered the fact that $p \to q$, $\neg p \vee q$, and $\neg(p \wedge \neg q)$ are equivalent. It is always the case that two compound statements are equivalent if and only if they

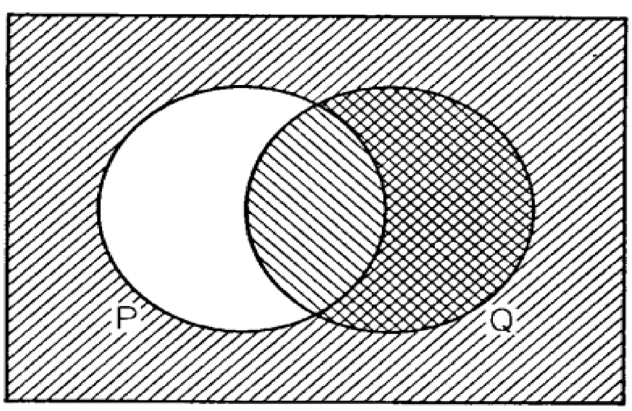

Figure 2.7: $\diamondsuit$

have the same truth sets. Thus we can test for equivalence by checking whether they have the same Venn diagram.

Suppose that p is a statement that is logically true. What is its truth set? Now p is logically true if and only if it is true in every logically possible case, so that the truth set of p must be $\mathcal{U}$. Similarly, if p is logically false, then it is false for every logically possible case, so that its truth set is the empty set $\emptyset$.

Finally, let us consider the implication relation. Recall that p implies p if and only if the conditional $p \to q$ is logically true. But $p \to q$ is logically true if and only if its truth set is $\mathcal{U}$, that is, $(\widetilde{P - Q}) = \mathcal{U}$, or $(P - Q) = \emptyset$. From Figure 2.4 we see that if $P - Q$ is empty, then P is contained in Q. We shall symbolize the containing relation as follows: $P \subset Q$ means "P is a subset of Q". We conclude that $p \to q$ is logically true if and only if $P \subset Q$.

Figure 2.8 supplies a "dictionary" for translating from statement language to set language, and back. To each statement relative to a set of possibilities $\mathcal{U}$ there corresponds a subset of $\mathcal{U}$, namely the truth set of the statement. This is shown in lines 1 and 2 of the figure. To each connective there corresponds an operation on sets, as illustrated in the next four lines. And to each relation between statements there corresponds a relation between sets, examples of which are shown in the last two lines of the figure.

Example 2.3 Prove by means of a Venn diagram that the statement $[p \vee (\neg p \vee q)]$ is logically true. The assigned set of this statement is $[P \cup (\tilde{P} \cup Q)]$, and its Venn diagram is shown in Figure 2.9. In that figure the set P is shaded vertically, and the set $\tilde{P} \cup Q$ is shaded

Statement Language	Set Language
r	R
s	S
$\sim r$	$\tilde{R}$
$r \vee s$	$R \cup S$
$r \wedge s$	$R \cap S$
$r \rightarrow s$	$\widetilde{(R - S)}$
r implies s	$R \subset S$
r is equivalent to s	$R = S$

Figure 2.8: ◇

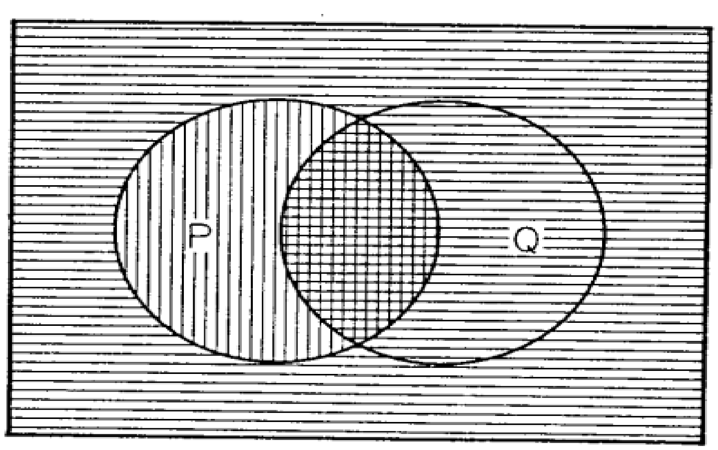

Figure 2.9: ◇

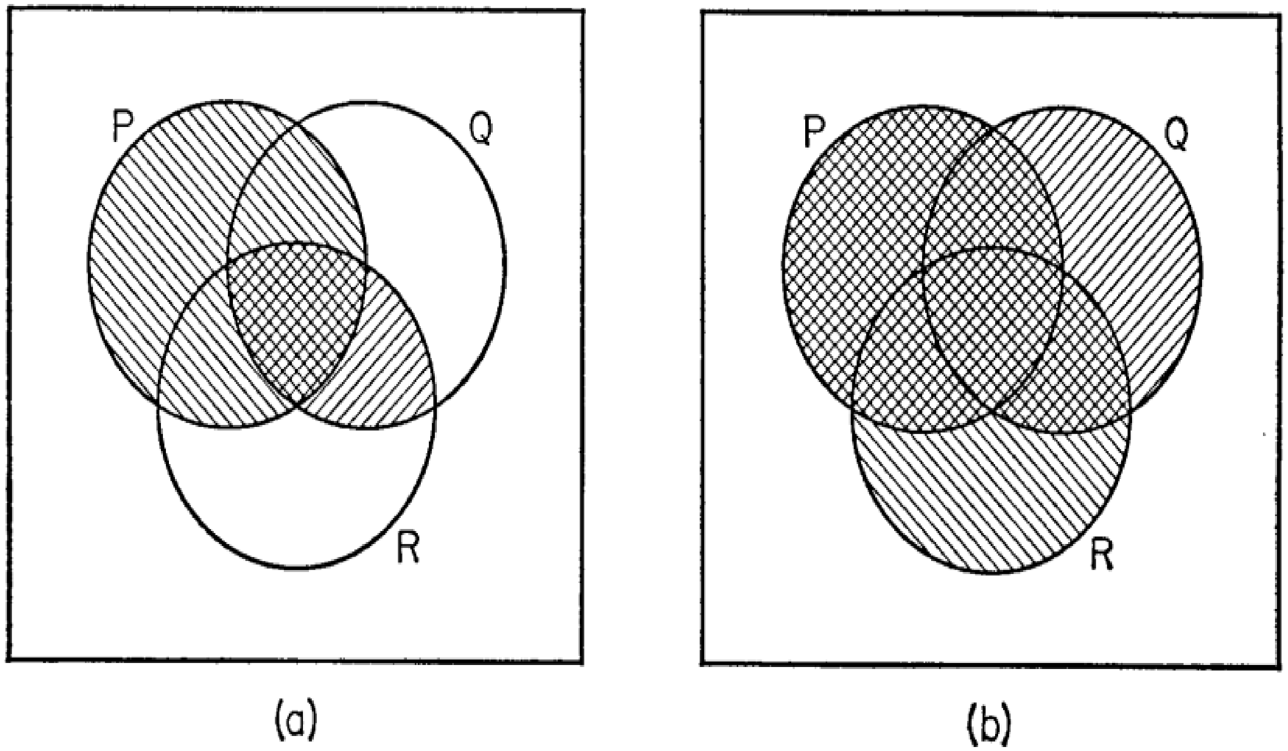

Figure 2.10: ♢

horizontally. Their union is the entire shaded area, which is $\mathcal{U}$, so that the compound statement is logically true. ♢

Example 2.4 Prove by means of Venn diagrams that $p \vee (q \wedge r)$ is equivalent to $(p \vee q) \wedge (p \vee r)$. The truth set of $p \vee (q \wedge r)$ is the entire shaded area in diagram (a) of Figure 2.10, and the truth set of $(p \vee q) \wedge (p \vee r)$ is the doubly shaded area in diagram (b). Since these two sets are equal, we see that the two statements are equivalent. ♢

Example 2.5 Show by means of a Venn diagram that q implies $p \to q$. The truth set of $p \to q$ is the shaded area in Figure 2.7. Since this shaded area includes the set Q. we see that q implies $p \to q$. ♢

The abstract laws of set operations

The set operations which we have introduced obey some very simple abstract laws, which we shall list in this section. These laws can be proved by means of Venn diagrams or they can be translated into statements and checked by means of truth tables.

The abstract laws given below bear a close resemblance to the elementary algebraic laws with which the student is already familiar. The resemblance can be made even more striking by replacing $\cup$ by $+$ and $\cap$ by $\times$. For this reason, a set, its subsets, and the laws of combination of subsets are considered an algebraic system, called a *Boolean algebra*—after the British mathematician George Boole who was the first person to study them from the algebraic point of view. Any other system obeying these laws, for example, the system of compound statements studied in Chapter **??**, is also known as a Boolean algebra. We can study any of these systems from either the algebraic or the logical point of view.

Below are the basic laws of Boolean algebras. The proofs of these laws will be left as exercises.

The laws governing union and intersection:

A1. $A \cup A = A$.
A2. $A \cap A = A$.
A3. $A \cup B = B \cup A$.
A4. $A \cap B = B \cap A$.
A5. $A \cup (B \cup C) = (A \cup B) \cup C$.
A6. $A \cap (B \cap C) = (A \cap B) \cap C$.
A7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
A8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
A9. $A \cup \mathcal{U} = \mathcal{U}$.
A10. $A \cap \emptyset = \emptyset$.
A11. $A \cup \emptyset = A$.
A12. $A \cap \mathcal{U} = A$.

The laws governing complements:

B1. $\tilde{\tilde{A}} = A$.

B2. $A \cup \tilde{A} = \mathcal{U}$.

B3. $A \cap \tilde{A} = \emptyset$.

B4. $\widetilde{A \cup B} = \tilde{A} \cap \tilde{B}$.

B5. $\widetilde{A \cap B} = \tilde{A} \cup \tilde{B}$.

B6. $\tilde{\mathcal{U}} = \emptyset$.

The laws governing set-differences:

C1. $A - B = A \cap \tilde{B}$.

C2. $\mathcal{U} - A = \tilde{A}$.

C3. $A - \mathcal{U} = \emptyset$.

C4. $A - \emptyset = A$.

C5. $\emptyset - A = \emptyset$.

C6. $A - A = \emptyset$.

C7. $(A - B) - C = A - (B \cup C)$.

C8. $A - (B - C) = (A - B) \cup (A \cap C)$.

C9. $A \cup (B - C) = (A \cup B) - (C - A)$.

C10. $A \cap (B - C) = (A \cap B) - (A \cap C)$.