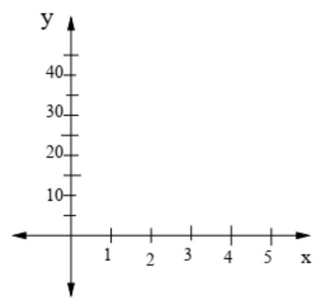
Straight Lines and Linear Functions

In this chapter we will learn about lines - how to draw them and how to find the equations. We will also be introduced to many important business models in this chapter that we will use this semester. Finally we will learn how to find the equation of a line if we have more than two data points.

The Cartesian Coordinate System

You should read this section. You do not need to find the distance between two points. You DO need to make sure you can draw appropriate axes for a math problem and plot pairs of points. When drawing axes for a problem, you should first determine what will be on the horizontal axis (usually x, but it could be the number of toasters or something else) and what will be on the vertical axis (usually y, but it could be the profit or something else). The next step is to determine the scale for each axis. Often they will be different. You may need to let x go from -10 to 10, or we may have up to 10 million toasters. You need to always keep in mind that we do not have a negative number of items! Next you need to determine the scale for the y-axis. Finally, you need to draw the axes, LABEL the axes and label the TICK marks. You will lose credit for unlabeled axes and unlabeled tick marks every time you graph. On the other hand, you will get credit for a set of labeled axes even if there is nothing graphed on them. You will have at least told me something!

A properly labeled set of axes should look like this:



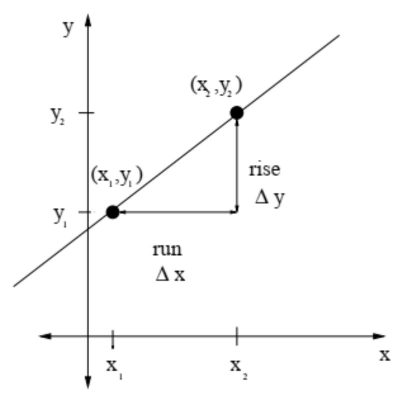
Straight Lines

Many important models are linear - that is, the graph of the model is a straight line. In this class we will use only linear equations, so you need to be sure you can handle them properly. In this section we will learn (review?) about the slope and expressing the equation of a line. We will also start using some important models that we will see most of the semester.

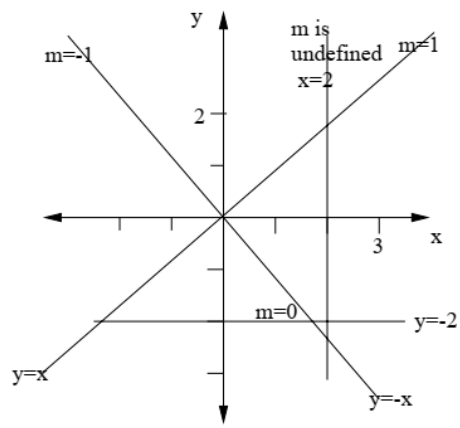
SLOPE: A vertical line has NO SLOPE. All other lines have a slope given by the equation

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$$

We can see this in a picture - the slope is a ratio of how the y changes as x changes:



We can also look at a picture of positive, negative, zero and no slope. Notice that the horizontal line (zero slope) is an equation of the form $y = c$. The y value doesn't change as x changes. Notice that the vertical line (no slope) is an equation of the form $x = c$. That is, the x value stays the same, no matter what y is.



A common method to express the equation of a line is the POINT-SLOPE form. This is useful when you know the slope of the line and a point on the line.

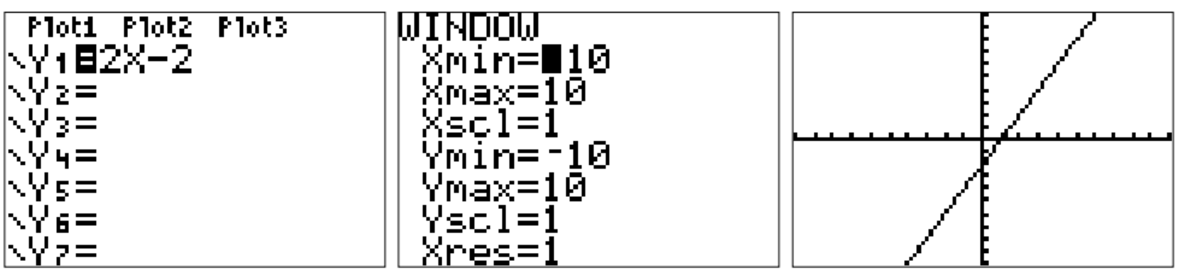
$$y - y_1 = m(x - x_1)$$

example - we are given that a line has a slope of 2 and goes through the point (3,4). what is the equation of the line?

answer - we will use the point-slope equation with $m = 2$, $x_1 = 3$ and $y_1 = 4$.

$$y - 4 = 2(x - 3) \rightarrow y = 4 + 2x - 6 = 2x - 2$$

You see that the initial form when we use the equation is not too useful, a bit of algebra puts it in the form to look at with our calculator, $y = 2x - 2$:



There is often some confusion on setting the window to view an equation. One thing to guide you is that we generally want to see the intercepts on the screen.

The x-intercept is where the line crosses the x-axis (that is, where $y = 0$).

The y-intercept is where the line crosses the y-axis (that is, where $x = 0$).

example - where are the intercepts for the line $y = 2x - 2$?

answer - to find the x-intercept I would set $y = 0$,

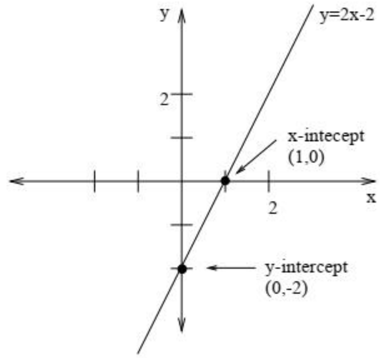
$$0 = 2x - 2 \rightarrow 2x = 2 \rightarrow x = 1$$

so the x-intercept is at (1,0). to find the y-intercept I would set $x = 0$,

$$y = 2 \cdot 0 - 2 \rightarrow y = -2$$

so the y-intercept is at (0,-2). You want to be sure that these two points are in your window.

You also need to remember that I do not see your calculator screen. I will be looking at what you write on your paper. If you were to turn in this problem with the line graphed at the intercepts found it should look like:



The form we generally like to express our line in is called the SLOPE-INTERCEPT form. This form uses the slope, m, and the y-intercept, b:

$$y = mx + b$$

This is also the form that your calculator uses.

A line can also be written in the GENERAL form, $Ax + By + C = 0$, but this is less useful that the other two forms.

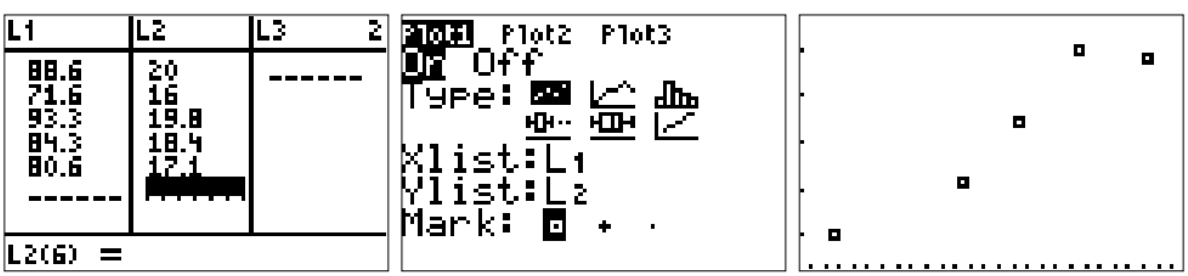
Two lines are parallel if their slopes are equal. So to determine if two lines are parallel, it is best to write them both in the slope-intercept form and find if the slopes are the same. If they have the same slope, they are parallel. If they also have the same intercept, they are the same line! Two lines are perpendicular if the product of their slopes is -1,

$$m_1 \cdot m_2 = -1$$

Application: We are given the following data about crickets:

temperature (°F)	88.6	71.6	93.3	84.3	80.6
chirps per second	20.0	16.0	19.8	18.4	17.1

We can graph this data in the calculator using the LIST function and and the STAT PLOT. Start by entering the data. Push the STAT button. then hit ENTER to use the edit mode. You can then put your data into the lists. I will put the temperature in the L1 list and the chirps per second in the L2 list. To graph this data we will use STAT PLOT (located above the y= button on the top row). Push 2nd then STAT PLOT and that puts you in the stat plot menu. we will use plot 1, so just hit ENTER to modify plot 1. We want to turn it on, so move the cursor to ON and hit ENTER. You also want to be sure to have the scatter plot with L1 as the Xlist and L2 as the Ylist. To graph it, I will first clear Y1=. The easiest way to get a good window with a stat plot is to go to ZOOM and number 9 is ZoomStat. this will automatically make a window that will fit the data. so scroll down the ZOOM menu to 9 and hit ENTER. (or just hit 9 in the ZOOM menu, if you remember it's number)



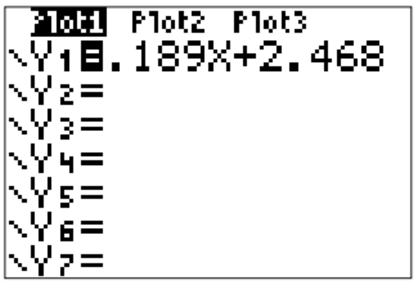
We can see that this sort of looks like a line. We can choose two points to make a *trend line*. We can then use the trend line to predict how many chirps per second when the temperature is at 68°. Choose a pair of points, say (71.6, 16) and (84.3, 18.4). We then want to find the equation of this line. We can use the point-slope form after we find the slope:

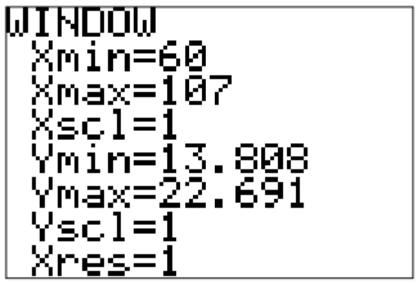
$$m = \frac{\Delta y}{\Delta x} = \frac{16 - 18.4}{71.6 - 84.3} \approx .189$$

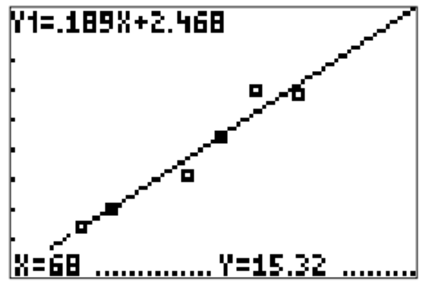
we can now choose either of the points to put into our equation,

$$y - 16 = .189(x - 71.6) \rightarrow y = 16 + .189x - 13.532 = .189x + 2.468$$

We can enter this as Y1. We will need to change the window so that we can see where 68F is. We can find the number of chirps as 68F by entering it into the formula, or tracing our line and finding the value at that point. If we want to find values at nice integers, we must be extra careful setting up our window. The screen of the calculator is 94 pixels wide. so we must use multiples of 94 to have a good screen for tracing. If we want to look from 60F on up, we can go from Xmin = 60 to Xmax = 60 + 9.4*5 = 107 (the calculator will work this out for you, you can type it in as an arithmatic problem). The TI-83 has a VERY nice feature to find the y window values. If you go to ZOOM now and scroll all the way down there is a ZoomFit at number 0. if you hit this button it will set a screen that fits your function values that you have chosen for the x's into the screen. (it may take a second or two).







To find the value at $x = 68$ we can hit the trace button. At first this will put us on the stat plot, but hit the up arrow to move to the Y1 function. Then you can move with the left and right arrows until you find X=68. We see that there would be about 15.3 chirps per second at this temperature. Or, if you are only interested in a single value, you can do CALC and then ENTER on 1 (VALUE) and it will go back to the graph and prompt with X= and then put in the value you want to find Y at and hit ENTER. Again it comes back with 15.13... (please only use a reasonable number of decimal places!)

Notice that the trend line gives us an approximate model for the chirps per second as a function of temperature. Is this the best model? We were only using 2 of our 5 pieces of data.

Linear Functions and Mathematical Models

A linear model is a model that is a linear function, $f(x) = mx + b$. We call x the independent variable and y the dependent variable. With our model we want to be able to pick a value for x and find what y will be. We will study several linear models in this section. You must be aware of the domain and range of your function. The domain is the x values that can put put into the function. Since we will only do linear functions (no worry about square roots or division by zero!) what you need to keep track of is the constraints on x introduced by the problem. When we are talking about the number of items produced, that number will never be negative. Therefore, when you graph the function you do not graph negative values of x. The range of the function is the y value that comes out. Usually as long as your x is in its proper domain, there is no concern.

Application - DEPRECIATION. One model that is exactly linear is depreciation. An item has an initial value and a final value and then it is assumed that the value decreases linearly with time (if the IRS uses this model, it must be good). We can use a car as an example. A car is purchased for \$15,000 and is kept for 5 years and at the end of the 5 years it is worth \$5000. Find an expression for the value of the car as a function of time. What is the car worth after 4 years? What is the rate of depreciation of the car?

We have a point representing the initial value of the car, (0, 15000) and the final value of the car (5, 5000). We find the slope of the line passing through these two points,

$$m = \frac{5000 - 15000}{5 - 0} = -2000$$

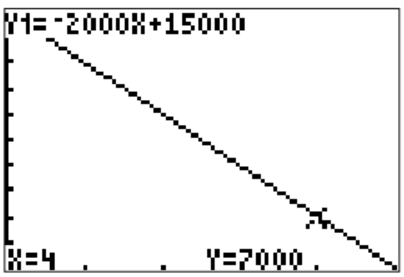
this is the rate of depreciation - the car decreases in value by \$2000 each year. We can find the equation,

$y - 15000 = -2000(x - 0) \rightarrow y = -2000x + 15000$

or to use the proper variable names for this kind of problem,

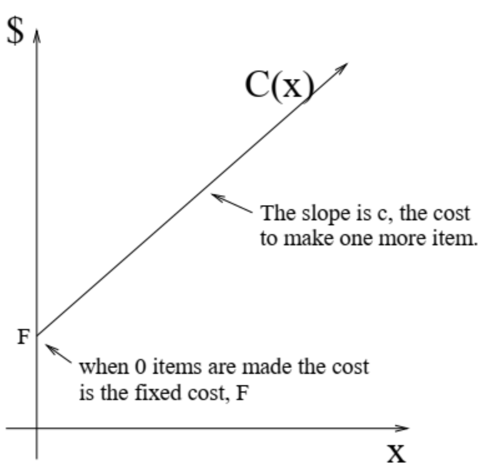
$V = -2000t + 15000$

where x (or t) is the number of years the car is kept. The value after 4 years is $V = -2000 * 4 + 15000 = 7000$ or \$7000. If you graph this function, be sure to show the line only where it is applicable, that is the domain is between $x = 0$ and $x = 5$ Also, if the problem is stated in terms of dollars, please remember to use dollars in the final answer too. We will not use units TOO much, however you should be careful to give your final answer in the proper units or lose credit.



Application - COST. Many important business models concern the cost of manufacturing an item. We will always write the cost of producing x items as $C(x)$. In a linear cost model the cost function has two parts. The *fixed costs*, F, are those costs that are independent of the number of items produced. These costs can include the cost of the management salaries, the real estate taxes, the maintanence costs. The *variable costs* are those costs that vary as the number of items produced. These costs include the cost of materials and labor. A typical cost equation looks like:

$$C(x) = cx + F$$



Example - A company makes heaters. They find that the cost to make 10 heaters is \$1500 and the cost to make 20 heaters is \$1900. Find the cost equation.

Answer - We are not given the fixed or variable costs, only the TOTAL cost to produce a certain number of heaters. If we look at our model, the slope of the cost equation is the variable cost. We are given two points on the cost line, so we can find the slope (or variable cost). You need to be sure to remember that the number of items is the x value!!!! So I will first write out our two points on the line being careful to put each number in the right place. The pairs of points are in the form $(x, C(x))$: (10, 1500) and (20, 1900). Now we find the slope

$$\text{variable cost} = m = \frac{1500 - 1900}{10 - 20} = \frac{-400}{-10} = 40$$

So the items cost \$40 each to make. To find the fixed cost we will use the point-slope form of the line;

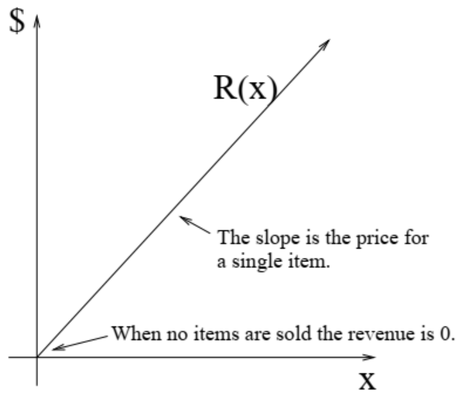
$$y - y_1 = m(x - x_1) \rightarrow y - 1500 = 40(x - 10) \rightarrow y = 1500 + 40x - 400 = 40x + 1100$$

So our fixed costs are \$1100 and our cost equation is

$$C(x) = 40x + 1100$$

It is a good idea to check after deriving an equation if it works. That is, if I put in $x = 10$, the cost should come out to be \$1500. And if I put in $x = 20$, the cost should come out to be \$1900. This will catch many silly mistakes.

Application - REVENUE. If x items are sold for s dollars each, the the money brought in, or revenue, from the sale of these items is $R(x) = sx$. This is a linear revenue model where the price of the item is a constant.

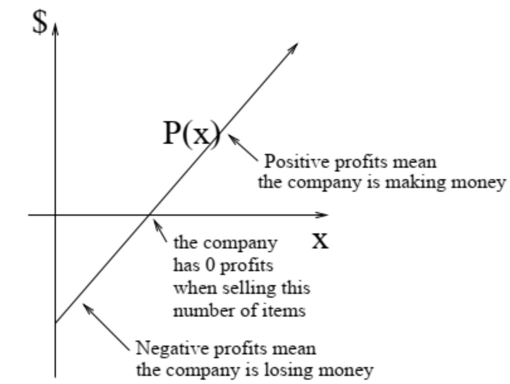


Example - What is the revenue from selling heaters if the heaters sell for 50 dollars each?

Answer - $R(x) = 50x$

Application - PROFIT. The profit made from selling x items is $P(x)$ and it is the difference between the revenue (money in) and the cost (money out).

$$P(x) = R(x) - C(x)$$



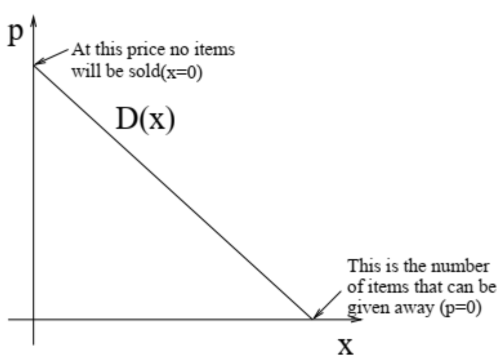
Example - What is the profit equation from selling x heaters?

Answer - We subtract the cost of making the heaters from the revenue brought in:

$$P(x) = 50x - (40x + 1100) = 10x - 1100$$

Watch out that you subtract the entire cost equation!

Application - DEMAND. We will use p for the price of an item (be sure to write carefully so it looks different than P). The demand (or price paid) for x items is $D(x) = p$. Generally the demand function looks like a line with a negative slope. That is, the there will be fewer items sold if the price is higher.



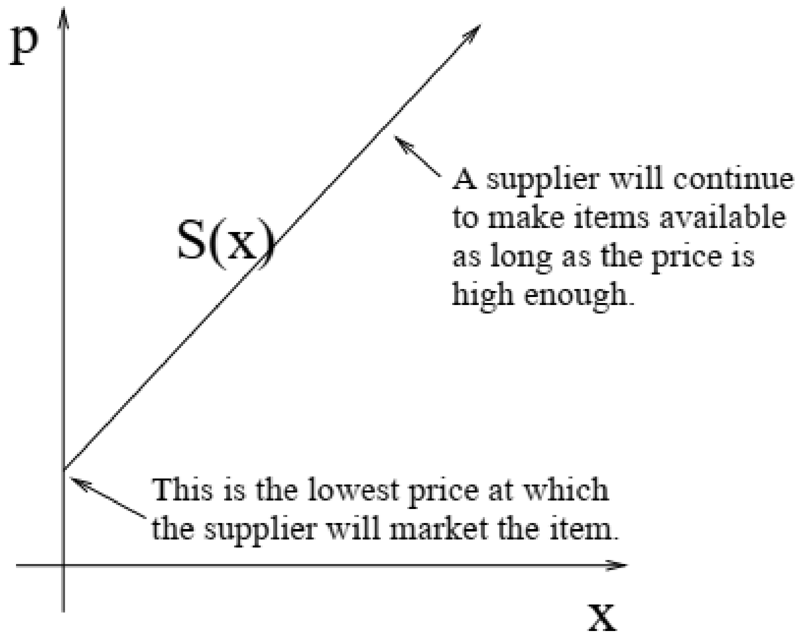
Example - Say a store finds that it can sell 10 rollerblades when the price is $180 each and that is sells 50 rollerblades when the price is $100 each. Find the demand equation.

Answer - Again we have two pairs of points. We have the number of items as the x value and the price as the y value, (x, p): (10, 180) and (50, 100). We find the slope and then use the point-slope form to find

$$D(x) = p = 200 - 2x$$

(again you would check that when you put in your original data that it works)

Application - SUPPLY. The supply function $p = S(x)$ models the relationship between the price of an item and how many are supplied to the market. If a manufacturer can get a high price for an item, it will supply more items.



Example - A company manufactures rollerblades. The company is not willing to sell rollerblades unless it can get \$60 each. It will supply 10 rollerblades if it can get \$80 each. Find the supply equation.

Answer - We must again rewrite our word problem as a pair of points. The number of items is ALWAYS x and the price will again be in the y position. When the company wants at least \$60 for supplying rollerblades that means one of our points is (0,60) and the other is (10,80). With our two points we can turn the usual crank and find

$$S(x) = p = 60 + 2x$$

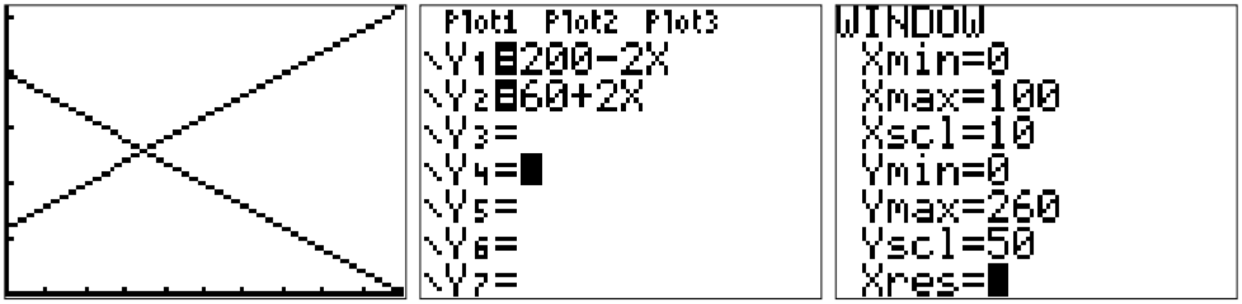
Intersection of Straight Lines

Finding the place that two lines intersect is a very important concept this semester. We will have several applications that simply require finding the intersection point, and later we will also need to find the intersection points of many lines to solve linear programming problems. So please take the time to understand this concept now.

Application - EQUILIBRIUM POINT. Is there a price that will satisfy the consumer and the producer? Let us plot the supply and demand equations on the same graph. If they intersect, that will be at a price and number of items that the producer and consumer agree upon. It is called the equilibrium point.

Example - What is the equilibrum point for the sale of rollerblades from the previous section?

Answer - Let's start by graphing the two lines. I will put them in as Y1 and Y2. To help determine the window I will look at the x-intercepts. The x-intercept (remember, this is from letting $y = 0$) of the supply equation is negative, so it will not help. x-intercept of the demand equation is 100. So I will try a window from XMIN=0 to XMAX=100 and then use ZOOMFIT to get the Y dimensions.



We see that the lines do cross. We label the intersection point as (x_0, p_0). **This point satisfies both equations at the same time!** x_0 is the equilibrium quantity and p_0 is the equilibrium price. How do we find these values? There are several methods to finding the place where two lines intersect and you may use whichever you prefer. I will not check you method, only your answer.

algebraic method: we have the pair of equations

$$\begin{aligned} p_0 &= D(x_0) = 200 - 2x_0 \\ p_0 &= S(x_0) = 60 + 2x_0 \end{aligned}$$

This may be solved by letting $p_0 = p_0$ or

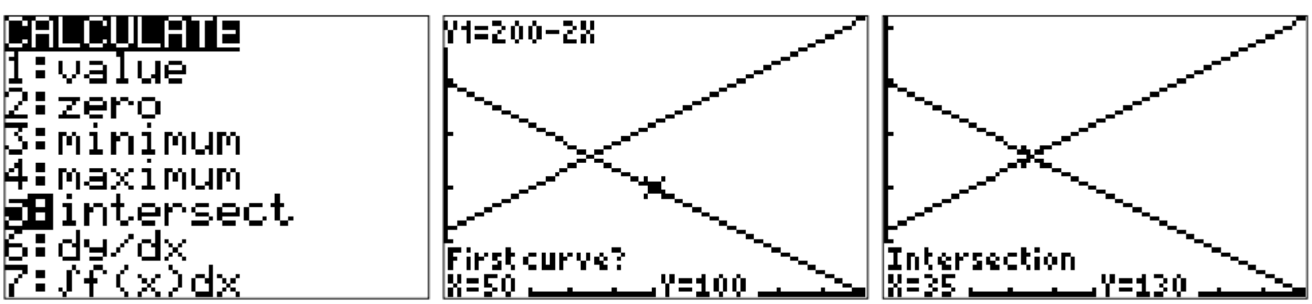
$$200 - 2x_0 = 60 + 2x_0 \rightarrow 140 = 4x_0 \rightarrow x_0 = 35$$

This tells us that the equibrium quantity is 35, but what is the price? We can put this x value back into either of the equations to find p_0,

$$p_0 = D(35) = 200 - 2 \cdot 35 = 130 = S(35) = 60 + 2 * 35 = 130$$

So the price that will satisfy the consumer and produce is \$130.

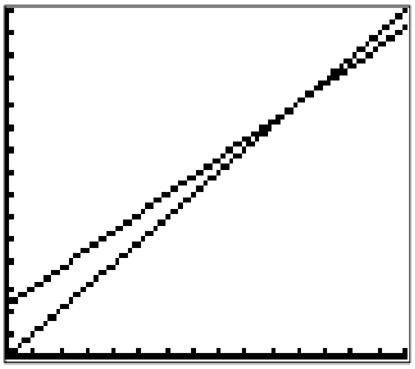
calculator method: Push CALC and then 5:intersect and ENTER. It will prompt you for the first curve. Check at the top of the screen if you are one of the two curves. If not, use the up or down arrow to move which curve you are on. When you are on the correct line, hit ENTER. Then you will be prompted for the second curve. Again use the up or down arrows to change curves. Hit ENTER when you are on the other line. Then it will prompt for a guess. you can use the left or right arrow to move near the point of intersection. Then hit ENTER. It will list the intersection point. The only caution on this method is that it will give the intersection point in decimal form and sometimes you need it in fraction (exact) form. Be very careful when you see a decimal!

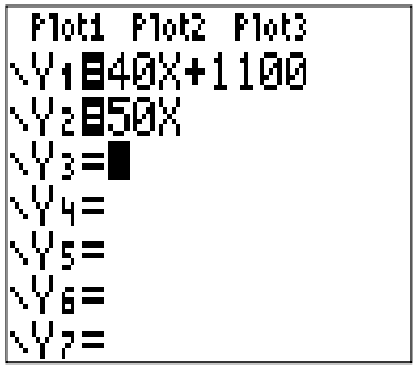


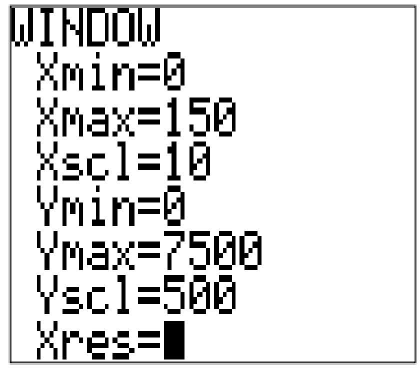
Application - BREAK-EVEN ANALYSIS. In a break-even analysis you are looking for the place where a company's cost is equal to their revenue, that is, they make no money. One way to find the break-even point is to look at the cost equation and the revenue equation and see where they are equal (intersect)

Example - What is the break-even point for the company making heaters? We found in the last section that the cost for making the heaters was $C(x) = 40x + 1100$ and the revenue equation was $R(x) = 50x$.

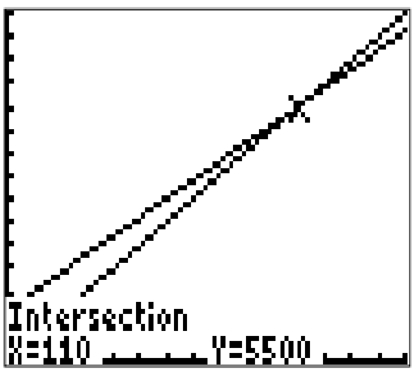
Answer - Start by graphing the two equations. The revenue equation and the cost equation both have positive slopes, so we will just have to play around a bit to find a screen that shows the intersection point. I tried some values and decided that XMIN=0 and XMAX=150 works with the ZOOMFIT to find the Y values:







To find the intersection I will try the calculator first:



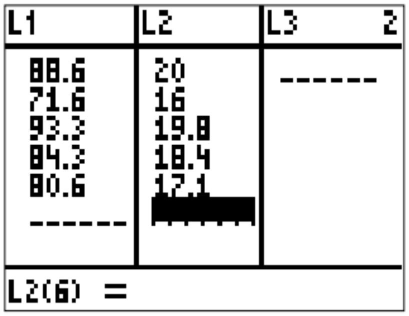
So the equilibrium point is (110, 5500). That means that the company does not make any money when they sell (make) 110 heaters, they just equal the costs (revenue) of \$5500. We can try putting this point into the profit equation and find

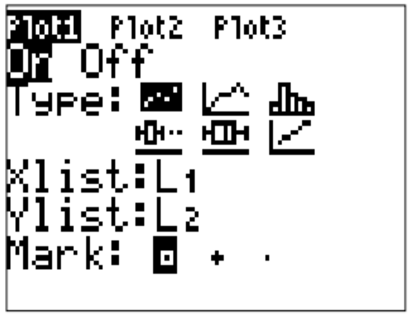
$$P(x) = R(x) - C(x) = 10x - 1100 \rightarrow P(110) = 10 \cdot 110 - 1100 = 0$$

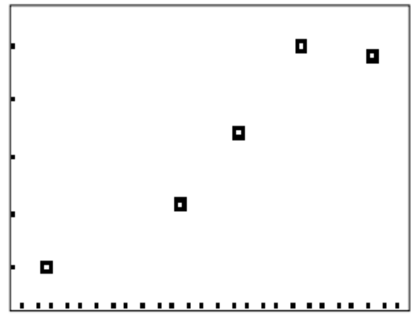
Break-even means zero profit.

The Method of Least Squares

In our example with the number of cricket chirps as a function of temperature we found a trend line using only two of the data points. Let us look at what we found:







What we would like to do is have a line that takes all of the data into account. We would like it to come as close as possible to all the data points. The line that has the smallest sum of the distance from all the points is the least squares (or regression) line. We will find this on the calculator. The book explains the way to find the regression line by hand, but we will NEVER do it that way!

You can refer to your manual for how to find least squares on your calculator. If you have a TI-83 or TI-83 Plus, there is a program LEASTSQR that will guide you through the procedure.

To run the LEASTSQR program, choose the PRGM button and select the LEASTSQR program. This will put you back to the home screen with prgmLEASTSQR on the screen. Hit ENTER (programs never have arguements). You can then select the option you want.

We can use the equation of the regression line to estimate what would happen for points that we do not have data for.

example - If the temperature was 75F, how many cricket chirps would you expect to hear?

answer - Y=16.5, so expect to hear about 16.5 chirps per second

example - If you counted 19 chirps per second, what was the temperature?

answer - X=87.39, so the temperature is about 87.4 degrees.