

Lecture 1

Learning Objectives

At the end of this class, students should be able to:

- solve system of linear equations
- plot graph of linear equation

1.1 Linear Equations

An equation is a statement of an equality containing one or more variables. A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and (the first power of) a single variable. For example, $3x + 5 = 9$, $2x + 3y = 12$ and $ax + by + cz = d$ are linear equations.

Solving the equation consists of determining which values of the variables make the equality true. An unknown in the given equation whose value has to be found is known as variable and the value of this variable which satisfies the equality is known as solution. For example, x is the variable in the equation $x + 2 = 5$. In this equation, if we put $x = 3$ then the equation will be satisfied. So $x = 3$ is the root or solution of the equation.

1.2 Equivalent Equations

Two equations are said to be equivalent, if they have exactly the same solutions. Solving an equation may involve performing operations on it. If we perform an operation on an equation that produces another equation with the same solution set, then the two equations are said to be equivalent. The basic idea in solving equations is to perform operations that produce simpler equivalent equations and to continue the process until we obtain an equation with an obvious solution. The following three operations provide equivalence equations.

1. Adding (subtracting) the same polynomial to (from) both sides of an equation, where the Polynomial is in the same variable as that occurring in the equation. For example, if we add $4x$ on both side of the equation $-2x = 6 - 4x$, we get $-2x + 4x = 6 - 4x + 4x$ or $2x = 6$ which is the equivalent equation with the given equation.
2. Multiplying (dividing) both sides of an equation by the same nonzero constant. For example, if we divide both side of the equation $2x = 6$ by 2 we get $x = 3$ which is the equivalent equation with the given equation.
3. Replacing either side of an equation by an equal expression. For example, if the equation is $x(x + 2) = 3$, then replacing the left side by the equal expression $x^2 + 2x$ gives the equivalent equation $x^2 + 2x = 3$.

1.3 Linear Equations in One Variable

A first-degree or linear equation in variable x has the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$. For example, $2x + 5 = 11$ is a linear equation in variable x .

Theorem: If $a \neq 0$, then the equation $ax + b = 0$ has precisely one solution, $x = -b/a$. For example, the solution of equation $2x - 7 = 0$ is $x = 7/2$.

Illustration 1

Solve $2(p + 4) = 7p + 2$

Solution

We have $2(p + 4) = 7p + 2$

or $2p + 8 = 7p + 2$

or $2p - 7p = 2 - 8$

or $-5p = -6$

or $5p = 6$

or $p = 6/5$

Illustration 2

A day care center's total monthly revenue from the care of x children is given by $R = 450x$, and its total monthly costs are given by $C = 380x + 3500$. How many children need to be enrolled each month to break even? In other words, when will revenue equal costs?

Solution

Total monthly revenue (R) = $450x$

Total monthly costs (C) = $380x + 3500$

For breakeven, $R = C$

i.e. $450x = 380x + 3500$

or $450x - 380x = 3500$

or $70x = 3500$

or $x = 3500/70 = 50$

Thus 50 children need to be enrolled in a month for breakeven.

1.4 Linear Equation in Two Variables

A linear equation involving two variables x and y has the standard form $ax + by = c$, where a , b , and c are real numbers and a and b cannot both equal to zero.

Illustration 3

Given the equation $2x + 3y = 6$.

- i) Determine any pair of values that satisfies the equation.
- ii) Determine the pair of values which satisfies the equation when $x = 0$.
- iii) Determine the pair of values which satisfies the equation when $y = 1$.

Solution

i) If we put $x = 1$ in the given equation, we get

$$2 \times 1 + 3y = 6$$

$$\text{or } 3y = 4$$

$$\therefore y = 4/3$$

Thus, one pair of values satisfying the equation is $x = 1$ and $y = 4/3$.

ii) Substituting $x = 0$ in the given equation, we get

$$2 \times 0 + 3y = 6$$

$$\text{or } 3y = 6$$

$$\therefore y = 2$$

When $x = 0$, the pair of values satisfying the equation is $x = 0$ and $y = 2$.

iii) Substituting $y = 1$, in the equation, we get

$$2x + 3 \times 1 = 6$$

$$\text{or } 2x = 3$$

$$\therefore x = 3/2$$

When $y = 1$, the pair of values satisfying the equation is $x = 3/2$ and $y = 1$.

1.5 Graph of a Linear Equation

We use two real number lines to form a Cartesian or rectangular coordinate system, one horizontal and one vertical; and let them cross through their origins as given in the following figure. These two number lines are called the horizontal axis and the vertical axis, or, together, the coordinate axes. The horizontal axis is usually referred to as the x axis and the vertical axis as the y axis, and each is labeled accordingly. The coordinate axes divide the plane into four parts called quadrants, which are numbered counterclockwise from I to IV.

We use Cartesian coordinates to pinpoint where we are on a graph. Using Cartesian Coordinates, we mark a point on a graph by how far along and how far up it is. The left-right (horizontal) direction is commonly called x . The up-down (vertical) direction is commonly called y .

In the following figure, for point P , the vertical line intersects the horizontal axis at a point with coordinate a , and the horizontal line intersects the vertical axis at a point with coordinate b . These two numbers, written as the ordered pair (a, b) form the coordinates of the point P . Thus we can say, the ordered pair (a, b) represents the point P that is located a units along the x -axis and b units along the y -axis. The first coordinate, a , is called the abscissa of P ; the second coordinate, b , is called the ordinate of P .

Similarly, the abscissa of Q is -5 and the ordinate of Q is 5 . The coordinates of a point can also be referenced in terms of the axis labels. The x coordinate of R is 10 , and the y coordinate of R is -10 . The point with coordinates $(0, 0)$ is called the origin.

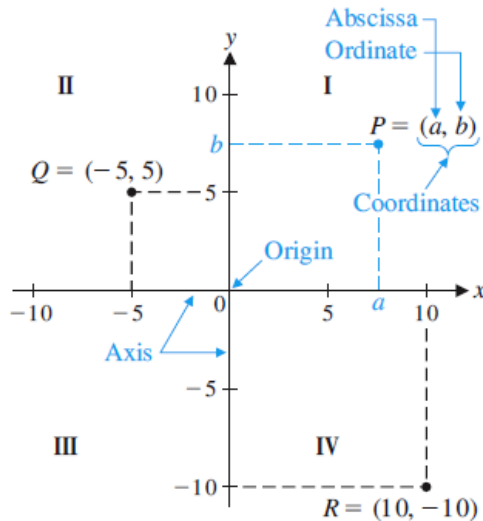


Illustration 4

Plot the graph of the equation: $2x + 4y = 8$.

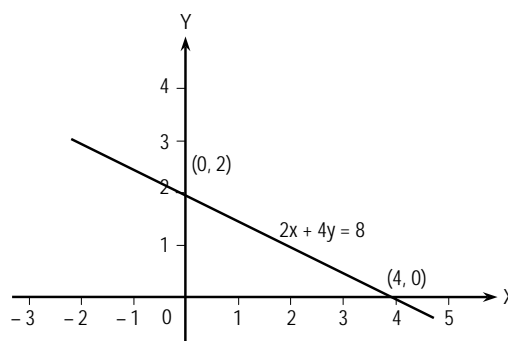
Solution

The given equation $2x + 4y = 8$ represents a linear equation in two variables. The graph of this equation is a straight line. The graph of this equation is found by first identifying any two pairs of values for x and y which satisfy the equation.

Note: *Aside from the case where the right side of the equation equals 0, the easiest points to identify are those found by setting one variable equal to 0 and solving for the value of the other variable. That is, let $x = 0$ and solve for the value of y ; again, let $y = 0$ and solve for the value of x .*

When we put $x = 0$ in the equation $2x + 4y = 8$, the corresponding value for y is 2, and when $y = 0$, the corresponding value of x is 4.

Thus $(0, 2)$ and $(4, 0)$ are two members of the solution set. The graph of the line passes through these two points. A straight line has connected the two points, and the line has been extended in both directions.



Note: The graph of the equation $x = a$ is the vertical line which crosses the x -axis at $x = a$. Similarly, The graph of the equation $y = b$ is the horizontal line which crosses the y -axis at $y = b$.

Illustration 5

Plot the graph of equation $x = 2$.

Solution

The graph of the equation $x = 2$ is a straight line passing through the point $(2, 0)$ and parallel to y -axis.

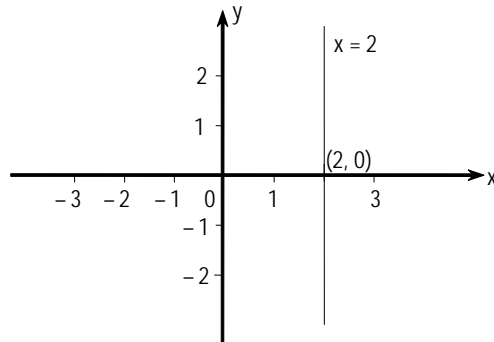
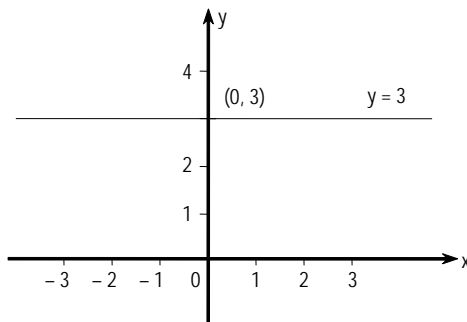


Illustration 6

Plot the graph of the equation $y = 3$.

Solution

The graph of the equation $y = 3$ is a straight line passing through the point $(0, 3)$ and parallel to x -axis.



Intercepts

The intercept of a line is the distance from the origin to the point where the line cuts the axis. Algebraically, the x -intercept of an equation $px + qy = r$ represents the values of x when $y = 0$. Similarly, the y -intercept of an equation is the point where the graph of the equation cuts the y -axis. Algebraically, the x -intercept represents the values of x when $y = 0$ and the y -intercept represents the values of y when $x = 0$. For example, x and y intercepts of the equation $2x + 4y = 8$ are 4 and 2 respectively.

The graph of a linear equation is a straight line, only two points are necessary to determine its graph. We can use intercepts to plot the graph of a linear equation.

Note: A linear equation may have

- i) x -intercept
- ii) y -intercept

iii) both x and y intercepts

iv) no intercept, for example, we do not get x and y intercepts for an equation $2x - y = 0$.

Slope

Any straight line, with the exception of vertical line, can be characterized by its slope. By ‘slope’ we mean basically the inclination of a line, whether it rises or falls as we move from left to right along the x -axis, and the rate at which the line rises or falls (in other words, how steep the line is).

The slope of a line may be positive, negative, zero, or undefined. A line with a positive slope rises from left to right, or runs uphill. Another way to describe such line is to say that the relationship between x and y is in such way that the value of y increases as x increases (or y decreases as x decreases). A line having a negative slope falls from left to right, or runs downhill. To state it differently, for such a line the value of y decreases as x increases (or y increases as x decreases). This means that x and y are behaving in an inverse manner: that is, as one increases, the other decreases and vice versa.

The slope tells us at what rate the value of y changes relative to changes in the value of x or the slope indicates the change in the value of y if x increases by 1 unit.

The slope (m) of the straight line connecting two points having coordinates (x_1, y_1) and (x_2, y_2) respectively, is computed as

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_1 \neq x_2.$$

Illustration 7

Find the slope of the straight line passing through the points $(2, 5)$ and $(4, 9)$.

Solution

$$\text{Here, slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 5}{4 - 2} = 2$$

Exercise for Reader

1. Solve the following equations.

a) $2(x - 7) = 5(x + 3) - x$

b) $\frac{5x}{2} - 4 = \frac{2x - 7}{6}$

c) $\frac{2x}{x - 3} = 4 + \frac{6}{x - 3}$

2. Suppose that the relationship between a firm’s profit P and the number x of items sold can be described by the equation $5x - 4P = 1200$

- a) How many units must be produced and sold for the firm to make a profit of \$150?
 - b) Solve this equation for P in terms of x .
 - c) Find the profit when 240 units are sold.
3. Plot the graph of the following equations.
- a) $2x + 3y = 12$
 - b) $x - 2y = 0$
 - c) $x - 2y = 5$

Lecture 2

Learning Objectives

At the end of this class, students should be able to:

- solve linear inequalities
- solve (2×2) system of equations by graphical method

2.1 Linear Inequalities

Inequalities express the condition that two quantities are not equal. For example, we write $5 < 8$ because 5 is to the left of 8 on the real number line. We may also say that 8 is greater than 5 (written as $8 > 5$). If the number a is less than or equal to another number b then it is written as $a \leq b$. We write $x \geq 5$ to indicate that x is greater than or equal to 5. Similarly, if x represents the number of desks a firm produces and sells, if at most 10 desks can be sold in a month then this can be written mathematically as $x \leq 10$.

Solving an inequality means finding its solution set, and two inequalities are equivalent if they have the same solution set. As with equations, we find the solutions to inequalities by finding equivalent inequalities from which the solutions can be easily seen. We use the following properties to reduce an inequality to a simple equivalent inequality.

2.2 Inequality Properties

An equivalent inequality will result, and the sense or direction will remain the same if each side of the original inequality

1. has the same real number added to or subtracted from it.
2. is multiplied or divided by the same positive number.

An equivalent inequality will result, and the sense or direction will reverse if each side of the original inequality

3. is multiplied or divided by the same negative number.

These properties can be explained as follows:

Let a , b , and c are real numbers.

1. If $a > b$ then $a + c > b + c$ and $a - c > b - c$.
2. If $a > b$ and c is a positive real number, then $ac > bc$ and $a/c > b/c$.
3. If $a > b$ and c is a negative real number, then $ac < bc$ and $a/c < b/c$.

Note: Multiplication by 0 and division by 0 are not permitted.

Thus, we can perform the same operations on inequalities that we perform on equations, with the exception that the sense of the inequality reverses if we multiply or divide both sides by a negative number. Otherwise, the sense of the inequality does not change.

2.3 Linear Inequality in One Variables

An inequality is a statement that one quantity is greater than (or less than) another quantity. The inequality $2x - 3 < x + 7$ is an example of first-degree (linear) inequality in one variable that states that the left member is less than the right member. Certain values of the variable will satisfy the inequality. These values form the solution set of the inequality. For example, if we start with the following statement

$$2x - 3 < 5x + 9$$

Adding 3 on both sides, we get

$$2x < 5x + 12$$

Subtracting $5x$ from both sides, we get

$$-3x < 12$$

Dividing both sides by -3 , we get

$$x > -4$$

Thus, $x > -4$ is the solution of the given inequality.

Illustration 1

Solve the inequality $\frac{4(x-3)}{2} \geq x+5$.

Solution

We have $\frac{4(x-3)}{2} \geq x+5$

or $4(x-3) \geq 2(x+5)$

or $4x - 12 \geq 2x + 10$

or $4x - 2x \geq 10 + 12$

or $2x \geq 22$

or $x \geq 11$

Thus, $x \geq 11$ is the solution of the given inequality.

Illustration 2

Ram can spend at most \$900 for a video camera and some video tapes. He plans to buy the camera for \$695 and tapes for \$5.75 each. Write an inequality that could be used to find the number of tapes (x) that he could buy. How many tapes could he buy?

Solution

According to question, the cost price for a video tape is \$5.75. Therefore, the cost price for x number of video tapes will be $\$5.75x$.

Now the total cost price of a camera and video tapes = $\$5.75x + \695

According to question, Ram can spend at most \$900, therefore, the required inequality is $\$5.75x + \$695 \leq \$900$

or $\$5.75x \leq \$900 - \$695$

or $\$5.75x \leq \205

or $x \leq 205/5.75 = 35.65$

Thus, Ram can buy at most 35 video tapes.

2.4 Linear Inequality in Two Variables

A linear inequality in two variables is a linear expression that involves any of the relational symbols such as $<$, $>$, \leq or \geq . For example, $x + y > 2$, $2x + 3y \leq 30$ are examples of linear inequalities as powers of x and y are 1. For an inequality to be linear, the power of variables should be 1 only. When the power of variables is more than 1, it becomes a non-linear inequality.

2.5 System of Linear Equations

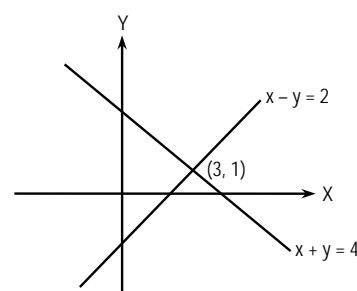
A system of equations is a set consisting of more than one equation. In solving system of equations, we are interested in identifying values of the variables that satisfy all equations in the system at the same time. For this reason, the group of equations that we solve is often called simultaneous system of equations. Here, we shall study two different techniques: graphical and elimination for finding solution of a system. While solving for the values of the variables which satisfy all the equations, we will check whether there exists any common point in the lines representing the equations.

2.6 Graphical Solution Procedure

Here, we talk about (2×2) system of equations, i.e., system of equations represented by two straight lines in two dimensions. For (2×2) system equations three different types of solution sets might exist. It is difficult to plot the graph of the equation in three variables, so we shall only discuss about the elimination procedure in that case.

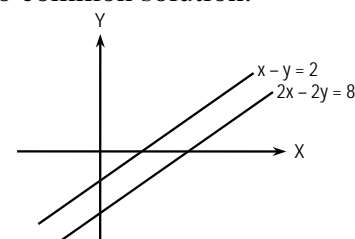
1. When the two lines graphed in a plane, if they intersect, or cross one other, the coordinates of the point of intersection represents the solution for the equations represented by these two lines. When there is just one pair of values for the variables that satisfy the system of equations, the system is said to have a unique solution.

For example, if we solve the equations: $x + y = 4$, and $x - y = 2$, we get unique solution $x = 3$ and $y = 1$. The graph is shown in the figure.



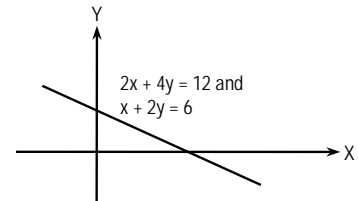
2. When the two lines are parallel to each other, i.e., they never meet, there will be no set of values of x and y both equations. In other words, they have no common solution.

For example, if we solve the equations $x - y = 2$, and $2x - 2y = 8$, we get no solution. The equations like $x - y = 2$, and $2x - 2y = 8$ are called inconsistent system of equations. The graph is shown in figure the figure.



3. The last possibility is the case when the graph of both the equations coincides, i.e., one fits exactly on the other. In this case, the set of values of x and y which satisfies the first equation satisfies the second also. In this case, we will get infinite number of solutions. The graph is as follows:

For example, if we solve the equations: $2x + 4y = 12$, and $x + 2y = 6$, we get infinite number of solution. The graph is shown in the figure.



Another way of summarizing the three conditions is as follows:

In (2×2) system of linear equations let m_1 and m_2 represents the respective slopes of the two lines and d_1 and d_2 represents the respective y intercepts.

- There is a unique Solution to the system if $m_1 \neq m_2$.
- There is no Solution to the system if $m_1 = m_2$, but $d_1 \neq d_2$.
- There are infinite number of Solutions to the system if $m_1 = m_2$ and $d_1 = d_2$.

Exercise for Reader

1. Solve the following inequalities.

a) $\frac{2x}{3} > 4 - x$

b) $\frac{4x}{5} - \frac{1}{6} < x - \frac{2(x+1)}{3}$

2. Solve the following system of equations by the graphical procedures.

a) $3x + 4y = 12$

$4x + 3y = 12$

b) $4x + 2y = 10$

$2x + y = 5$

c) $x + 2y = 5$

$x + 2y = 8$

Lecture 3

Learning Objectives

At the end of this class, students should be able to:

- solve (2×2) System of equations
- solve (3×3) System of equations

3.1 Elimination Procedure

In elimination technique, we apply algebraic operations to eliminate one or more variables from the given system in order to get the solution of the system.

(2×2) System

The eliminating procedure can be generalized as follows for a (2×2) system equations:

1. Multiply (if necessary) the equations by constants so that the coefficients on one of the variables are the negatives of one another in the two equations.
2. Add the two resulting equations. The following three situations may arise.
 - (a) If adding the equations results in a new equation having one variable, there is a unique solution to the system. Solve for the value of the remaining variable, and substitute this value back into one of the original equations to determine the value of the variable that was originally eliminated.
 - (b) If adding the equation results in an identity, i.e., an equation that is always true, such as $0 = 0$ or $4 = 4$, the two original equations are equivalent to each other and there are an infinite number of solutions to the system.
 - (c) If adding the equations results in a false statement, say, $0 = 5$, the equations are inconsistent and there is no solution set.

We can summarize the procedure in the following tree diagram

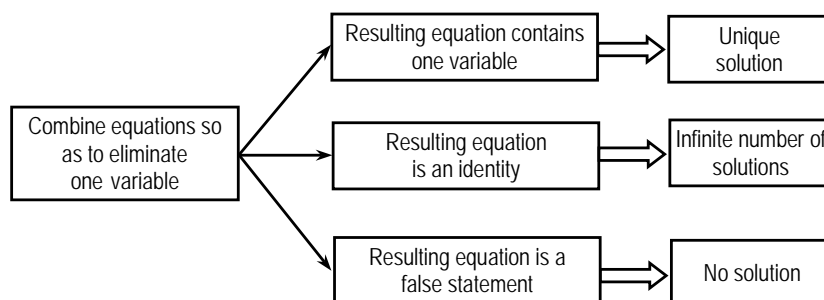


Illustration 1

Solve the following system of equations by elimination method:

$$2x + 5y = 11$$

$$3x + 4y = 13$$

Solution

We have

$$2x + 5y = 11 \quad \dots \text{(i)}$$

$$3x + 4y = 13 \quad \dots \text{(ii)}$$

Multiplying equation (i) by 3 and equation (ii) by -2 and then adding, we get

$$\begin{array}{r} 6x + 15y = 33 \\ -6x - 8y = -26 \\ \hline 7y = 7 \end{array}$$

$$\text{or } y = 1$$

Now substituting $y = 1$ in equation (i), we get

$$2x + 5 \times 1 = 11$$

$$\text{or } 2x = 11 - 5$$

$$\text{or } x = 3$$

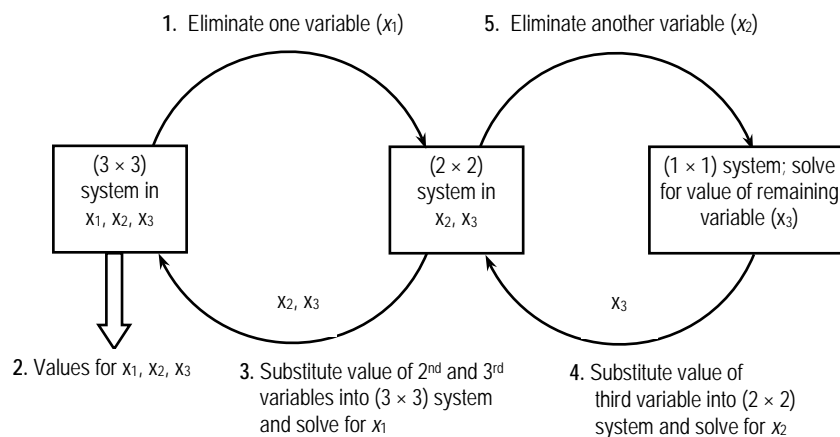
Thus, the required solution of the given system is $x = 3, y = 1$.

(3 × 3) System

The elimination procedure for (3×3) systems is similar to that of (2×2) systems. Here we start with the (3×3) system and try to reduce this to an equivalent system having two variables and two equations. After eliminating one of three variables, the same procedure as used for (2×2) systems is employed to eliminate a second variable, which results it in a (1×1) system.

After solving for the remaining variable, its value has to be substituted sequentially back through the (2×2) system and finally the (3×3) system to determine the values of the other two remaining variables.

We can summarize the procedure in the following figure.



Exercise for Reader

1. Solve the following system of equations by the elimination procedures.

a) $3x + 4y = 12$

$$4x + 3y = 12$$

b) $4x + 2y = 10$

$$2x + y = 5$$

c) $x + 2y = 5$

$$x + 2y = 8$$

2. Solve the following system of equations.

a) $x + y + z = 6$

$$2x - y + 3z = 4$$

$$4x + 5y - 10z = 13$$

b) $-2x + y + 3z = 12$

$$x + 2y + 5z = 10$$

$$6x - 3y - 9z = 24$$

c) $x + y + z = 20$

$$2x - 3y + z = -5$$

$$6x - 4y + 4z = 30$$

Lecture 4

Learning Objectives

At the end of this class, students should be able to:

- solve quadratic equations
- solve quadratic inequalities

4.1 Quadratic Equations

The general form of the equation is given by,

$$ax^2 + bx + c = 0, \text{ where } a \neq 0.$$

Let us identify its solutions by completing the square:

First, dividing both sides by a and putting the constant term in the right side, we get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\frac{b^2}{4a^2}$ to both sides of the equation, we get

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\text{or } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Extracting the square root, we get

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, we get two solutions $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ of the quadratic equation $ax^2 + bx + c = 0$.

Thus, two solutions (roots) of a quadratic equation $ax^2 + bx + c = 0$ can be found directly by applying the formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and this is known as quadratic formula.

Illustration 1

Solve the quadratic equation: $6x^2 + 7x - 10 = 0$

Solution

$$\begin{aligned}\text{Here, } x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 6 \times (-10)}}{2 \times 6} \\ &= \frac{-7 \pm \sqrt{289}}{12} \\ &= \frac{-7 \pm 17}{12}\end{aligned}$$

$$\text{Thus } x = \frac{-7-17}{12} \text{ or } x = \frac{-7+17}{12}$$

$$\text{i.e. } x = -2 \text{ or } x = \frac{5}{6}$$

Thus, the solutions of the given equation are -2 and 5/6.

Factoring Method

We can find the solution of a quadratic equation by using factoring method also. The following illustrations clarify the concept.

Illustration 2

Solve the equation: $x^2 + 12x + 36 = 0$.

Solution

We have $x^2 + 12x + 36 = 0$

$$\text{or } (x)^2 + 2 \times x \times 6 + (6)^2 = 0$$

$$\text{or } (x+6)^2 = 0$$

$$\text{or } x+6 = 0$$

Thus, $x = 6$ is the solution of given equation.

Illustration 3

Solve the equation: $x^2 + x - 56 = 0$

Solution

We have $x^2 + x - 56 = 0$

$$\text{or } x^2 + (8-7)x - 56 = 0$$

$$\text{or } x^2 + 8x - 7x - 56 = 0$$

$$\text{or } x(x+8) - 7(x+8) = 0$$

$$\text{or } (x+8)(x-7) = 0$$

$$\text{or } x = -8, 7$$

Thus, the solutions of the given equation are -8 and 7.

4.2 Quadratic Inequalities

When we have quadratic inequality, we can solve it by using factorization. The following illustrations provide solution procedure.

Illustration 4

Solve the inequality: $x^2 - 3x + 2 \leq 0$.

Solution

We have $x^2 - 3x + 2 \leq 0$

$$\text{or } (x - 1)(x - 2) \leq 0$$

The following attributes of the two factors on the left-hand side will result in the inequality being satisfied.

Condition	Factor		Product
	$(x - 1)$	$(x - 2)$	
I	$= 0$	Any value	0
II	Any value	$= 0$	0
III	> 0	< 0	< 0
IV	< 0	> 0	< 0

Condition I: $x - 1 = 0$ when $x = 1$

Condition II: $x - 2 = 0$ when $x = 2$

Condition III: $x - 1 > 0$ and $x - 2 < 0$
when $x > 1$ and $x < 2$

Condition IV: $x - 1 < 0$ and $x - 2 > 0$
when $x < 1$ and $x > 2$

From conditions I and II, we get $x = 1$ and $x = 2$. From condition III, we get $1 < x < 2$. There is no value of x which can satisfy condition IV.

Thus, from these conditions, the required solution of the given inequality is $1 \leq x \leq 2$.

Illustration 5

Solve the inequality: $x^2 - 4 \geq 0$.

Solution

We have $x^2 - 4 \geq 0$

$$\text{or } (x - 2)(x + 2) \geq 0$$

The following attributes of the two factors on the left-hand side will result in the inequality being satisfied.

Condition	Factor		Product
	$(x - 2)$	$(x + 2)$	
I	$= 0$	Any value	0
II	Any value	$= 0$	0
III	> 0	> 0	> 0
IV	< 0	< 0	> 0

Condition I: $x - 2 = 0$ when $x = 2$

Condition II: $x + 2 = 0$ when $x = -2$

Condition III: $x - 2 > 0$ and $x + 2 > 0$
when $x > 2$ and $x > -2$

Condition IV: $x - 2 < 0$ and $x + 2 < 0$
when $x < 2$ and $x < -2$

From conditions I and II, we get $x = 2$ and $x = -2$. Condition III suggests that $x > 2$, while condition IV has identified that $x < -2$.

Combining all these results, we get $x \geq 2$ and $x \leq -2$. Thus, the solution to the given inequality is $x \leq -2$ or $x \geq 2$.

Exercise for Reader

1. Solve the following quadratic equations.

a) $2x^2 - 7x + 3 = 0$

b) $2x^2 + x - 4 = 0$

c) $15x^2 - 28 = x$

d) $x - \frac{1}{x} = 3, \quad x \neq 0$

2. A two digits number is such that the product of digits is 20. When 9 is added to the number then the digits interchange their places. Find the number.

3. Solve the following inequalities.

a) $(x - 2)(x + 5) \geq 0$

b) $x^2 - 16 \leq 0$

Lecture 5

Learning Objectives

At the end of this class, students should be able to:

- represent the set
- plot Venn-diagram
- apply various set operations

5.1 Sets

A set is a well-defined collection of objects. Well defined means that there exists a rule with the help of which it is possible to decide if a given object belongs to that collection or not. Thus, every collection is not a set.

The following are the examples of the set:

- The set of government universities in a particular country.
- The set of students in a particular college.

The objects that belong to a set are called elements of the set. The elements of a set are denoted by small letters whereas the sets are denoted by capital letters.

We use the symbol \in (belongs to) to indicate the set membership, and non-membership is indicated by \notin (does not belong to). If an element x belongs to a set A , we use the notation $x \in A$. Similarly, the notation $y \notin A$ indicates that an object y is not a member of set A .

Two sets A and B are equal if both have the same elements. This is denoted as $A = B$. If they are unequal, it is expressed as $A \neq B$.

If every element of a set A is also an element of set B , then A is called a subset of B . This is also expressed as: A is contained in B , or B contains A . This is expressed symbolically as $A \subseteq B$. For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 4, 5, 6\}$, we can say $A \subseteq B$.

A subset X is a proper subset of a set Y if X is subset of Y and $X \neq Y$, written as $X \subset Y$. ϕ is a proper subset of any non-empty set. For example, let $X = \{1, 2, 5\}$ and $Y = \{1, 2, 3, 5, 7\}$ then we can say X is subset of Y ; moreover, X is a proper subset of Y .

According to the number of elements, sets can be put on two categories. Sets that contain definite number of elements are called finite sets. Sets that contain unlimited number of elements are called infinite sets.

A set which contains only one element is called singleton set. If there is no element in a set, it is called null or empty set, denoted by ϕ or $\{\}$.

A Universal Set is the set of all elements in a particular context, denoted by capital U .

5.2 Set Representation

A set may be represented in different ways. The following are most common methods of describing a set.

Listing Method

According to this method, the elements are listed, separated by commas and enclosed in braces $\{ \}$. For example, if we denote the set of letters in the word ‘mathematics’, then it is written as $M = \{m, a, t, h, e, i, c, s\}$.

Note that changing the arrangement of the elements of a set does not change the set. For example, sets $\{a, b, c\}$ and $\{b, a, c\}$ represent the same set. Also, repetition of elements of a set does not change the set.

This method is convenient when the number of elements in a set is small or when it is not easy or possible to articulate a property defining membership in the set.

Descriptive Method

According to this method, the set is defined by stating a property required for membership in the set. We use symbol to denote an arbitrary element of the set. For example, the set of letters of the word ‘mathematics’ can be written as

$$M = \{x: x \text{ is a letter of the word 'mathematics'}\}.$$

The 'x' to the left of the ':' indicates the general notation for an element of a set, the expression to the right states the condition(s) required of an element for membership in the set.

5.3 Venn Diagram

It is a pictorial representation of the relationship among the sets. Venn diagram utilizes rectangular and circular (or elliptical) areas to represent sets. A large rectangular (or square) area is used to designate the universal set and circular (or elliptical) areas inside the rectangle represent all other sets.

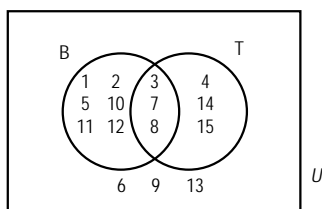
Illustration 1

Represent the following sets in the Venn diagram: $B = \{1, 2, 3, 5, 7, 8, 10, 11, 12\}$, $T = \{3, 4, 7, 8, 14, 15\}$, and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

Solution

Here, sets B and T are subsets and U is the universal set. We use circles to represent subsets and rectangle to represent universal set. There are few elements common to both the sets B and T, such sets are called intersecting sets. If two sets have no elements in common, they are said to be disjoint. Two sets which are disjoint can never have a superset-subset relationship, unless one is a null set.

The relationship between the given sets U, B, and T is shown in the following Venn diagram.



5.4 Set Operations

There are four basic set operations: union, intersection, difference, and complementation. The results of these operations are again sets.

Union of Sets

The union of two sets is a set that consists of all elements found in either set or both sets. The union of sets A and B is denoted by $A \cup B$, and defined by

$$\begin{aligned} A \cup B &= \{x: x \text{ belongs to } A \text{ or } B \text{ or both}\} \\ &= \{x: x \in A \text{ or } x \in B\} \end{aligned}$$

Illustration 2

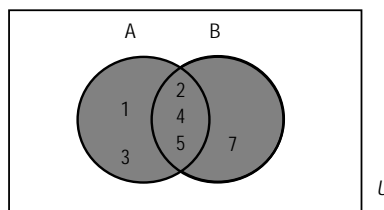
If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5, 7\}$, then find $A \cup B$ and represent it in a Venn diagram.

Solution

Since $A \cup B$ consists of all the elements that belong to either A or B or both, therefore,

$$A \cup B = \{1, 2, 3, 4, 5, 7\}$$

The shaded region in the following Venn diagram represents $A \cup B$.



Intersection of Sets

The intersection of two sets is the set consisting of all elements that belong to both the sets. The intersection of sets A and B is denoted by $A \cap B$, and defined by

$$A \cap B = \{x: x \in A \text{ and } x \in B\}$$

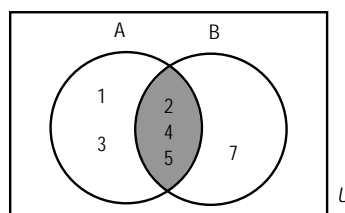
Illustration 3

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5, 7\}$, then find $A \cap B$ and represent it in a Venn diagram.

Solution

Since $A \cap B$ consists of all the elements that belong to A and B both, therefore, $A \cap B = \{2, 4, 5\}$

The shaded region in the following Venn diagram represents $A \cap B$.



Difference of Two Sets

The difference of two sets A and B is the set of all elements of A that do not belong to B , denoted by $A - B$, and defined by

$$A - B = \{x: x \in A, x \notin B\}$$

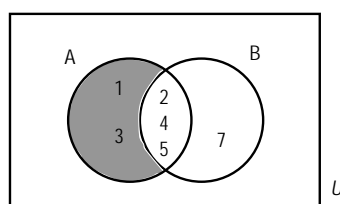
Illustration 4

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5, 7\}$, then find $A - B$ and represent it by a Venn diagram.

Solution

Since $A - B$ consists of all the elements that belong to set A but not B , therefore, $A - B = \{1, 3\}$.

The shaded region in the following Venn diagram represents $A - B$.



Complement of a Set

The complement of a set A is the set of all the elements of universal set which do not belong to A , denoted by A^C or A' , and defined by

$$A' = \{x: x \in U, x \notin A\}, \text{ or } A' = U - A$$

Illustration 5

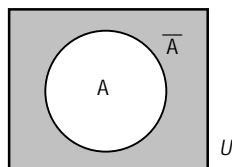
If $U = \{1, 2, 3, \dots, 9\}$ and $A = \{1, 3, 5\}$, then find A' and represent it in a Venn diagram.

Solution

Here $U = \{1, 2, 3, \dots, 9\}$ and $A = \{1, 3, 5\}$

Thus, $A' = \{2, 4, 6, 7, 8, 9\}$.

The shaded region in the following Venn diagram represents A' .



Exercise for Reader

Let the sets: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 5, 6\}$, $B = \{0, 2, 5\}$, and $C = \{3, 5, 6, 9, 10\}$. Then find:

- a) $B \cup C$ b) $A \cap B$ c) $A \cap (B \cup C)$ d) $A \cup (B \cap C)$ e) $A - B$
f) $\overline{A \cap A}$ g) $\overline{A \cap B}$ h) $\overline{A \cup B \cup C}$ i) $A \cap \overline{B} \cap \overline{C}$

Lecture 6

Learning Objectives

At the end of this class, students should be able to:

- identify the cardinality of particular set

6.1 Cardinality of Sets

Let $A = \{4, 7, 8, 12\}$. There are four elements in the set A . The number of elements contained in a set A is denoted by $n(A)$. Here, $n(A) = 4$. The cardinality of a set is the number of elements in a set.

Let $n(A)$, $n(B)$, $n(C)$, and $n(U)$ denote the number of elements contained in the sets A , B , C , and the universal set (U) respectively, then

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, where $n(A \cap B)$ denotes the number of elements of the set $A \cap B$.
2. $n_o(A) = n(A) - n(A \cap B)$, where $n_o(A)$ denotes the number of elements belonging to A but not B .
3. If A and B are disjoint sets, then $n(A \cup B) = n(A) + n(B)$
$$[\because A \cap B = \phi]$$
4. $n(A \cup B \cup C)$
$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$
5. $n_o(A) = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$, where $n_o(A)$ denotes the number of elements belonging to A but not B and C .

Illustration 1

Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 7\}$. Calculate the number of elements in the union $A \cup B$.

Solution

Here, $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 7\}$ then $A \cap B = \{2\}$.

Thus, $n(A) = 3$ and $n(B) = 4$ and $n(A \cap B) = 1$.

We know that, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

i.e. $n(A \cup B) = 3 + 4 - 1 = 6$

Hence, there are 6 elements in the union.

Illustration 2

Fifty people are interviewed about their food preferences. Twenty of them like Greek food, 32 like Italian food, and 12 like neither Greek nor Italian food.

- a) How many like both Greek and Italian?

b) How many like Greek but not Italian food?

Solution

Here, we have $n(U) = 50$, $n(G) = 20$, $n(I) = 32$, and $n(G \cup I)' = 12$.

a) We know that $n(G \cup I) = n(U) - n(G \cup I)' = 50 - 12 = 38$

Now applying, $n(G \cup I) = n(U) + n(G) - n(G \cap I)$, we get

$$38 = 20 + 32 - n(G \cap I)$$

or $n(G \cap I) = 14$

Hence, there are 14 people who like both Greek and Italian food.

b) In order to find the people who like Greek but not Italian food, we apply

$$\begin{aligned} n_o(G) &= n(G) - n(G \cap I) \\ &= 20 - 14 = 6 \end{aligned}$$

Hence, there are 6 people who like Greek but not Italian food.

Illustration 3

Of the cars sold during the month of July, 90 had air conditioning, 100 had automatic transmissions, and 75 had power steering. Five cars had all three of these extras. Twenty cars had none of these extras. Twenty cars had only air conditioning; 60 cars had only automatic transmissions; and 30 cars had only power steering. Ten cars had both automatic transmission and power steering.

- a) How many cars had both power steering and air conditioning?
- b) How many had both automatic transmission and air conditioning?
- c) How many cars were sold in July?

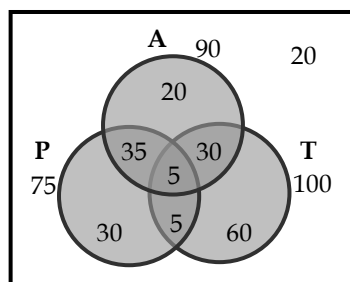
Solution

Let us denote the set of cars having air condition by A, cars having automatic transmission by T, cars having power steering by P, then we have

$$n(A) = 90, n(T) = 100, n(P) = 75, n(A \cap T \cap P) = 5, n(A \cup T \cup P)' = 20,$$

$$n_o(A) = 20, n_o(T) = 60, n_o(P) = 30, n(T \cap P) = 10.$$

The given information is presented in the following Venn diagram.



From the Venn diagram,

$$n(P) = n_o(P) + n(P \cap A) + n(P \cap T) - n(A \cap T \cap P)$$

i.e. $75 = 30 + n(P \cap A) + 10 - 5$

or $n(T \cap A) = 40$

Hence, there were 40 cars which had both power steering and air conditioning.

Again,

$$n(T) = n_o(T) + n(T \cap A) + n(T \cap P) - n(A \cap T \cap P)$$

i.e. $100 = 60 + n(T \cap A) + 10 - 5$

or $n(T \cap A) = 35$

Hence, there were 35 cars which had both automatic transmission and air conditioning.

Now,

$$\begin{aligned} n(U) &= 20 + 35 + 5 + 30 + 30 + 5 + 60 + 20 \\ &= 205 \end{aligned}$$

Thus, 205 cars were sold in July.

Exercise for Reader

1. In a class containing 50 students, 25 study marketing, and 27 study finance; and 12 study both. How many students study
 - a) Marketing only?
 - b) Finance but not marketing?
 - c) Neither marketing nor finance?
2. A survey of 500 high school seniors was conducted to determine the extent to which child abuse exists. It was found that 50 respondents recalled having physically abused by their fathers, 60 recalled having been physically abused by their mothers, and 20 recalled having been physically abused by both parents. How many respondents were:
 - a) Physically abused?
 - b) Not physically abused?
3. Of a group of 200 students, 100 are interested in music, 70 are interested in photography, and 40 like swimming; furthermore 40 are interested in both music and photography, 30 are interested in both music and swimming, 20 are interested in both photography and swimming and 10 are interested in all three activities. Find the number of students that are interested in

- a) Photography but not music and swimming.
 - b) Music only.
 - c) Swimming only.
 - d) Exactly one activity.
 - e) At least one activity.
 - f) None of the activities.
4. Records at a small college show the following about the enrollments of 100 first-year students in mathematics, fine arts, and economics.

38 take math

42 take fine arts

20 take economics

4 take economics and fine arts

15 take math and economics

9 take math and fine arts

12 take math and economics but not fine arts

Draw a Venn diagram representing this information and label all the areas. Use this diagram to answer the following.

- a) How many take none of these three courses?
 - b) How many take math or economics?
 - c) How many take exactly one of these three courses?
5. In a survey of 940 workers, 680 owned cycle, 620 owned TV and 250 owned houses. Also, 380 owned cycles and TV both, 110 owned houses and TV, 120 workers owned cycles and houses but not TV and 70 owned all three.
- a) How many workers owned cycle alone?
 - b) How many workers owned houses alone?
 - c) How many workers did not own any one of three things?