Lecture 7

Learning Objectives:
At the end of this class, students should be able to:

- understand the concept of real number system
- familiar with absolute value
- solve the problems related to permutations
- solve the problems related to combinations

7.1 Real Numbers

We study about the following numbers before talking about real numbers.

Natural Numbers

The numbers 1, 2, 3, . . . that are used for counting are called natural numbers. The set of natural numbers is denoted by ***N***.
Thus, ***N*** = {1, 2, 3, . . .}.

Whole Numbers

If we include 0 in the set of natural numbers it becomes set of whole numbers. The set of natural numbers is denoted by ***W***.
Thus, ***W*** = {0, 1, 2, 3, . . .}.

Integers

All the natural numbers including their negatives and zero are called integers. The set of integers is denoted by ***Z*** or ***I***.
Thus, ***Z*** = {…, -3, -2, -1, 0, 1, 2, 3, …}.

Rational Numbers

Rational numbers are simply numbers that can be written as fractions or ratios. A number which can be expressed in the form p/q, where p and q are integers and $q \neq 0$, is called a rational number. The set of rational numbers is denoted by ***Q***.
Thus, ***Q*** = $\{p/q: \ p, q \in Z, q \neq 0\}$
For example, the numbers 1/2, 5/7, 9/4 etc. are rational numbers.

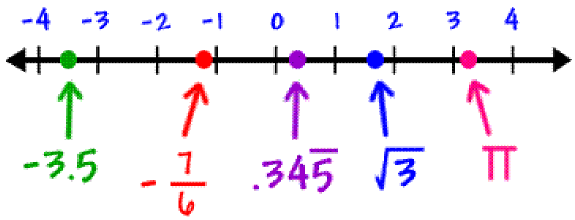
Irrational Numbers

An irrational number is a number that cannot be expressed as a fraction p/q for any integers p and q. Numbers represented by non-terminating and non-repeating decimals are also irrational numbers. Numbers such as $\sqrt{2}$ = 1.04142…, $\sqrt{3}$ = 1.732…are examples of irrational numbers. The set of irrational numbers is denoted by Q'.

Real Numbers

A collection of rational numbers and irrational numbers constitute the set of real numbers. It is denoted by ***R***. So, a real number is either rational or irrational; and ***R*** = $Q \cup Q'$.

Numbers can be represented in the number line as follows.



Prime and composite Numbers

A prime number is a whole number that only has two factors which are itself and one. A composite number has factors in addition to one and itself. The numbers 0 and 1 are neither prime nor composite. All even numbers are divisible by two and so all even numbers greater than two are composite numbers. All numbers that end in five are divisible by five. Therefore, all numbers that end with five and are greater than five are composite numbers.

7.2 Absolute Value

Let x be a real number. The absolute value (or modulus or numerical value) of x, written as $|x|$, is a non-negative number, defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

For example, $|6| = 6$, $|-3.5| = 3.5$ and $|0| = 0$, therefore, the absolute value of a number is always nonnegative, i.e., $|x| \geq 0$.

Thus, we can write $|x| = \max(x, -x)$

Properties of Absolute Values

1. Let x be any real number, then
   a) $|x| \geq 0$
   b) $|-x| = |x|$
   c) $-x \leq |x|$, and $x \leq |x|$

2. For any two real numbers x and y, then
   a) $|x-y| = |y-x|$
   b) $|x+y| \leq |x|+|y|$
   c) $|x-y| \geq |x|-|y|$
   d) $|x.y| = |x|.|y|$
   e) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $\quad (y \neq 0)$
3. If a is a positive real number then $|x| < a$ implies $-a < x < a$. Similarly, $|x| \leq a$ implies $-a \leq x \leq a$.
4. If a is a positive real number then $|x| > a$ implies either $x < -a$ or $x > a$. Similarly, $|x| \geq a$ implies either $x \leq -a$ or $x \geq a$.

7.3 Permutations and Combinations

Permutation is the rearrangement of objects or symbols where the order matters. Each unique ordering is called a permutation. Selections where the order does not matter and each object can be chosen only once are called combinations.

Fundamental Counting Principle

If there are *m* ways in which an event A can occur, and if there are *n* ways in which a second event B can occur after the first event has occurred, then the two events can occur in *mn* ways. Let us understand the concept with the help of following illustrations.

Suppose that you decide to dine at a restaurant that offers 6 courses: you might have 3 appetizer choices, 2 soup choices, and 4 salad choices, along with 5 main course choices, 10 beverage choices, and 3 dessert choices. How many unique 6-course meals you can make? We can answer this question by using the Fundamental Counting Principle. The total number of possible meals at the restaurant is $3 \times 2 \times 4 \times 5 \times 10 \times 3 = 3{,}600$ possible unique meals.

Illustration 1

Suppose there two roads connect cities A and B, four roads connect B and C, and five connect C and D. To drive from A, to B, to C, and then to city D, how many different routs are possible?

Solution

Here we have three stage procedure. The first (A→B) has two possibilities, the second (B→C) has four possibilities, and the third (C→D) has five possibilities. According to Fundamental Counting Principle, the total number of routs is $2 \times 4 \times 5 = 40$.

Permutations

The number of possible ordered arrangements of r objects chosen from a set of n objects is called the number of permutations of n objects taken r at a time, and it equals

$$^{n}P_{r} = \frac{n!}{(n-r)!} \qquad (r \le n)$$

If r = n, $^{n}P_{n}$ = n!

Illustration 2

A politician sends a questionnaire to her constituents to determine their concerns about six important national issues: unemployment, the environment, taxes, interest rates, national defense, and social security. A respondent is to select four issues of personal concern and rank them by placing the number 1, 2, 3, or 4 after each issue to indicate the degree of concern, with 1 indicating the greatest concern and 4 the least. In how many ways can a respondent reply to the questionnaire?

Solution

A respondent is to rank four of the six issues. Thus, we can consider a reply as an ordered arrangement of six items taken four at a time, where the first item is the issue with rank 1, the second is the issue with rank 2, and so on. Hence, this is a permutation problem, and the number of possible replies is

$$^{6}P_{4} = \frac{6!}{(6-4)!}$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$$

Illustration 3

Flynn, Peters, and Walters are forming an advertising firm and agree to name it by their three last names. How many names for the firm are possible?

Solution

Here, total number of firms (n) = 3,

Total number of last names (r) = 3

Possible number of advertising firms is $^3P_3 = \frac{3!}{(3-3)!}$

$$= \frac{3 \times 2 \times 1}{0!} \qquad = \frac{6}{1} \qquad = 6$$

Permutations with Repeated Objects

The number of distinguishable permutations of n objects such that n_1 are of one type, n_2 are of second type, . . . , and n_k are of k^{th} type, where $n_1 + n_2 + n_3 + \; . \; . \; . \; + n_k = n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

Illustration 4

Find the number of all possible words that can be formed using the letters of the word NINETEEN.

Solution

There are 8 letters of which N occurs 3 times, E occurs 3 times.

Hence, the required number of possible words

$$= \frac{8!}{3! \times 3!} \qquad = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3! \times 3 \times 2 \times 1}$$

$$= 1120$$

Permutations Involving Repetitions

The number of permutations of n different things taken r at a time, when each thing may be repeated up to r times in any arrangement, is n^r.

Illustration 5

How many five-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 if each digit may be used repeatedly in each number?

Solution

Here, n = 9 and r = 5 then

The possible number of five-digit numbers = $9^5 = 59049$.

Circular Permutation

The number of permutations of n different things taken all at a time, along a circle is (n – 1)!.

Illustration 6

In how many ways can 6 gentlemen be seated at a round table for the dinner?

Solution

Six gentlemen be seated at a round table in (6 – 1)! = 5! = 120 different ways.

Combinations

A selection of r objects, without regard to order and without repetition, selected from n distinct objects is called a combination of n objects taken r at a time. The number of such combinations is denoted by ^nC_r which is defined as

$$^nC_r = \frac{n!}{r!(n-r)!} \quad ; r \le n$$

Illustration 7

On a 13-question mathematics examination, a student must answer any 9 questions. In how many ways can the 9 questions be chosen (without regard to order)?

Solution

Here, n = 13 and r = 9 then

Out of 13 questions 9 can be selected in ${}^{13}C_9$ ways.

$$^{13}C_9 = \frac{13!}{9!(13-9)!}$$

$$= \frac{13 \times 12 \times 11 \times 10 \times 9!}{9! \times 4 \times 3 \times 2 \times 1}$$

$$= 13 \times 11 \times 5$$

$$= 715$$

Illustration 8

In a 10-question examination, each question is worth 10 points and is graded right or wrong. Considering the individual questions, in how many ways can a student score 80 or better?

Solution

According to question, a student has to solve 8 questions to score 80 marks. Thus, a student has to solve 8 or more questions to score 80 or better marks.

Thus, the number of different ways in which a student can score 80 or better

$$= {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$$

$$= 45 + 10 + 1$$

$$= 56$$

Exercise for Reader

1. Rewrite the following by using absolute sign
   a) $-2 \le x \le 8$
   b) $-4 < x < -2$
2. Rewrite the following without using absolute value sign:
   a) $|2x - 1| < 4$
   b) $|x + 3| \le 6$
3. At a restaurant, a complete dinner consists of an appetizer, an entree, and a beverage. The choices for the appetizer are soup and salad; for the entree, the choices are chicken, fish, steak, and lamb; for the beverage, the choices are coffee, tea, and milk. How many complete dinners are possible?
4. A university issues a questionnaire whereby each student must rank the four items with which he or she is most dissatisfied. The items are

tuition fees	parking fees
dormitory rooms	professors
cafeteria food	class sizes

   The ranking is to be indicated by the numbers I, 2, 3 and 4, where 1 indicates the item involving the greatest dissatisfaction and 4 the least. In how many ways can a student answer the questionnaire?
5. How many six-letter words from the letters in the word MEADOW are possible if no letter is repeated?
6. How many four-digit numbers can be formed from the digits 1, 3, 5, 7, 8, and 9
   a) if each digit may be used once in each number?
   b) if each digit may be used repeatedly in each number?
7. If a club has 20 members, how many different four-member committees are possible?
8. A quality-control technician must select a sample of 10 dresses from a production lot of 74 couture dresses. How many different samples are possible?
9. A college promotion committee consists of five members. In how many ways can the committee reach a majority decision in favor of a promotion?