Lecture 8

Learning Objectives:
At the end of this class, students should be able to:

- understand the concept of function
- identify domain and range of function
- plot the graphs of functions

8.1 Functions

A function *f* is a rule that assigns to each value of *x* one and only one value of *y*. The number *y* is normally denoted by $f(x)$, read '*f* of *x*,' emphasizing the dependency of *y* on *x*.

For example, let *x* and *y* denote the radius and area of a circle, respectively. The relationship between the area of a circle and its radius given by $y = \pi x^2$. This equation defines *y* as a function of *x*, since for each possible value of *x* (a nonnegative number representing the radius of a certain circle) there corresponds precisely one number $y = \pi x^2$ giving the area of the circle. This area function may be written as $f(x) = \pi x^2$.

In functional relationship $y = f(x)$, the variable *x* is known as the *independent* variable, and the variable *y* is called the *dependent* variable. The set of all values that may be assumed by *x* is called the *domain* of the function *f*, and the set comprising all the values assumed by $y = f(x)$ as *x* takes on all possible values in its domain is called the *range* of the function *f*.

For the area function $f(x) = \pi x^2$, the domain of *f* is the set of all nonnegative numbers *x*, and the range of *f* is the set of all nonnegative numbers *y*.

The output $f(x)$ associated with an input *x* is unique. For example, if *x* denotes an item in a department store and *y* denotes the corresponding selling price of that item. Then, each *x* must correspond to one and only one *y*. However, the different *x*'s may be associated with the same *y*. That means, the different items may have the same price.

8.2 Determining the Domain of a Function

To determine the domain of a function, we need to find what restrictions, if any, are to be placed on the independent variable *x*. In many practical applications, the domain of a function is dictated by the nature of the problem as given in the following illustration.

A \$360,000 building is depreciated by its owner. The value *y* of the building after *x* months of use is $y = 360{,}000 - 1{,}500x$. Thus, the value of the building is a function of the months used. Here, the variable of *x* in this problem can assume non-negative values only. Thus, the negative numbers are excluded from the domain of this function. The value of the building becomes zero when $x = 240$. Therefore, the maximum value of *x* can be 240. Thus, an independent variable *x* can assume the any value between 0 and 240. Therefore, the domain of this function is the interval [0, 240].

Illustration 1

Find the domain of the function $f(x) = x^2 + 5$.

Solution

Here, there is no restriction on the values of *x* because any real number *x* satisfies the given function. So the domain of *f* is the set of all real numbers, which is represented by the interval (-∞, ∞).

Illustration 2

Find the domain of the function $f(x) = \sqrt{x-3}$.

Solution

Since the square root of a negative number is undefined, therefore, it is necessary that $x - 3 \geq 0$. That is $x \geq 3$. So, the domain of *f* is the set of real numbers that are greater or equal to 3, which is represented by the interval [0, ∞).

Illustration 3

Find the domain of the function $f(x) = \dfrac{1}{x^2 - 25}$.

Solution

The only restriction on *x* is that $x^2 - 25$ must be different from zero since division by zero is not allowed.

Thus, the given function is not defined if $x^2 - 25 = 0$

i.e. $(x-5)(x+5) = 0$

i.e. $x = -5, 5$

Thus, the domain of the given function is the set of real numbers except 5 and -5, which is represented by the intervals (-∞, -5), (5, 5), and (5, ∞).

1.3 Graphs of Functions

Functions can be represented graphically. This graphical portrayal brings an added dimension to the understanding of mathematical functions. Functions which contain two variables (one dependent and one independent) are graphed on a set of rectangular axes. The vertical axis is used to represent the dependent variable and the horizontal axis is used to represent the independent variable.

To graph a function, we can simply assign different values from the domain for the independent variable and compute the corresponding value for the dependent variable. The resulting ordered pairs of values for the two variables specify the coordinates of the points which lie on the graph of the function.

To sketch the function, first we determine an adequate number of ordered pairs of values which satisfy the function. Then we locate their coordinates relative to a pair of axes. After that, we connect these points by a smooth curve to determine a sketch of the graph of the function.

Illustration 4

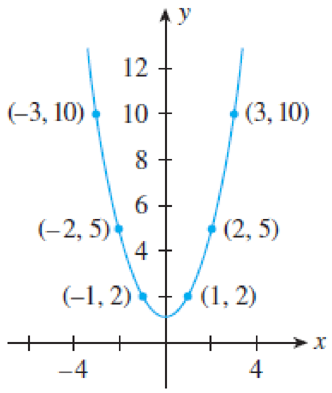
Sketch the graph of the function: $y = f(x) = x^2 + 1$.

Solution

Let us list the coordinates (*x*, *f*(*x*)) of some points on the graph of *f* in tabular form, as follows.

x	-3	-2	-1	0	1	2	3
f(*x*)	10	5	2	1	2	5	10

By plotting these points and then connecting them with a smooth curve, we obtain the graph of $y = f(x) = x^2 + 1$ as shown in the figure.



The domain of this function is the set of all real numbers. We can visualize this from the graph also.

To determine the range of this function, we observe that $x^2 \geq 0$ if x is any real number, and so $x^2 + 1 \geq 1$ for all real numbers x. Thus, the range of the function is $[1, \infty)$. We can visualize this result from the above graph.

Exercise for Reader

1. Let *f* be the function defined by $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$. Find $f(-2)$, $f(0)$, and $f(1)$.
2. Let *f* be the function defined by $f(x) = \sqrt{x+1}$. Find $f(3)$ and domain of *f*.
3. Find the domain of the following functions.
   a) $f(x) = \sqrt{5-x}$
   b) $f(x) = \dfrac{5}{x-2}$
   c) $f(x) = \sqrt{x^2 + 4}$