Lecture 12

Learning Objectives:

At the end of this class, students should be able to:

- understand the concept of logarithm
- familiar with logarithmic functions
- solve related problems
- understand the concept of break-even

12.1 Logarithm

A logarithm is the power to which a number must be raised in order to get some other number.

We know that

$$2^3 = 8$$

which is equivalent to

$$\log_2 8 = 3$$

read as log base 2 of 8 is 3.

For example, if $\log_4 16 = x$ or $4^x = 16$ then find the value of x. Here, $\log_4 16 = 2$ is equivalent to $4^2 = 16$. Therefore, the value of $x = 2$.

In general, $b^v = u \Leftrightarrow \log_b u = v$ for $b > 0$, $b \neq 1$ and u is a positive real number.

The b is called the base in both $v = \log_b u$ and $u = b^v$ and v is the *logarithm* in $v = \log_b u$ and the *exponent* in $u = b^v$. Thus, a logarithm is an exponent.

The two most widely used systems of logarithms are the system of common logarithms, which uses the number 10 as its base, and the system of natural logarithms, which uses the irrational number e = 2.71828. . . as its base.

The more common way of expressing $\log_{10} x$ is $\log x$. Similarly, the more common way of expressing $\log_e x$ is $\ln x$.

12.1 Logarithmic Properties

Computations involving logarithms are facilitated by the following laws of logarithms.

Let $b > 0$ and $b \neq 1$, and M and N are positive real numbers, then

1. $\log_b b = 1$
2. $\log_b 1 = 0$
3. $\log_b (b^x) = x$
4. $\log_b M = \log_b N$ if and only if $M = N$

5. $\log_b(MN) = \log_b M + \log_b N$
6. $\log_b(M / N) = \log_b M - \log_b N$
7. $\log_b(M^N) = N \log_b M$
8. $\log_b x = (\log_c x)/(\log_c b)$ where $b > 0$, $c > 0$, $b \neq 1$ and $c \neq 1$

Illustration 1

Solve the equation: $10^x = 12$.

Solution

We have $10^x = 12$

Taking base 10 logarithm on both sides, we get

$\log 10^x = \log 12$

or $x \log 10 = \log 12$

or $x \times 1 = \log 12$ [since $\log_{10} 10 = 1$]

or $x = 1.079$ [using calculator]

Thus, the solution to the given equation is $x = 1.079$.

12.3 Logarithmic Functions

The inverse of an exponential function is called a logarithmic function. If x is a positive number, then the logarithm of x to the base b (where, $b > 0$ and $b \neq 1$.), denoted by $\log_b x$, is the number y such that $b^y = x$; that is,

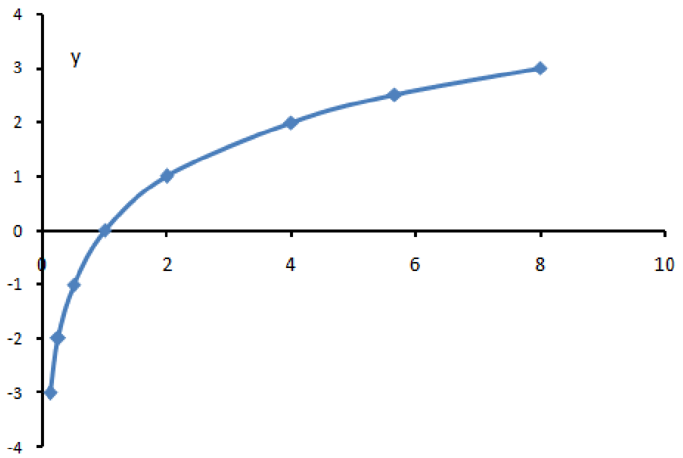
$$y = \log_b x \text{ if and only if } b^y = x \text{ for } x > 0$$

For example, $y = \log_2 x$ is a logarithmic function. Its graph is given in the following figure.

We can plot the graph of $y = \log_2 x$ by plotting the graph of $x = 2^y$.

$x = 2^y$	1/8	1/4	1/2	1	2	4	5.65	8
y	-3	-2	-1	0	1	2	2.5	3

The graph of the function $y = \log_2 x$ is as follows:



Basic Properties of the Logarithmic Function

The logarithmic function $y = \log_b x$ ($b > 0$ and $b \neq 1$) has the following properties.

1. The domain of the function is $(0, \infty)$.
2. The range of the function is $(-\infty, \infty)$.
3. The graph is a continuous curve on $(0, \infty)$.
5. The graph of the function $y = \log_b x$ increases as x increases.

Illustration 2

A community grows in such a way that t years from now, its population is $P(t)$ thousand, where $P(t) = 51 + 100 \ln(t + 3)$

a) What is the population when $t = 0$?

b) How long does it take for the population to double its initial value?

Solution

a) Here, $P(0) = 51 + 100 \ln(0 + 3)$

$= 51 + 100 \ln 3$

$= 160.861$

b) According to question $P(t) = 2P(0) = 2 \times 160.861$

Using $P(t) = 51 + 100 \ln(t + 3)$, we get

$2 \times 160.861 = 51 + 100 \ln(t + 3)$

or $270.722 = 100 \ln(t + 3)$

or $\ln(t + 3) = 2.70722$

or $t + 3 = e^{2.70722}$

or $t + 3 = 14.98$

or $t = 11.98$

It takes 11.98 years for the population to double its initial value.

12.4 Break-Even Analysis

In break-even analysis, the primary objective is to determine the break-even point. The break-even point may be expressed in terms of (1) volume of output, (2) total dollar sales, or possibly (3) percentage of production capacity.

Consider a firm with (linear) cost function $C(x)$, revenue function $R(x)$, and profit function $P(x)$ given by

$C(x) = cx + F$,

$R(x) = sx$, and

$P(x) = R(x) - C(x) = (s - c)x - F$

where c denotes the unit cost of production, s the selling price per unit, F the fixed cost incurred by the firm, and x the level of production and sales.

The level of production at which the firm neither makes a profit nor sustains a loss is called the breakeven level of operation.

At the level of production x_0, the profit is zero and so

$$P(x_o) = R(x_o) - C(x_o) = 0$$

or $$R(x_o) = C(x_o)$$

The point (x_0, y_0), is referred to as the break-even point; the number x_0 and the number y_0 are called the break-even quantity and the break-even revenue, respectively.

Illustration 3

Prescott manufactures its products at a cost of $4 per unit and sells them for $10 per unit. If the firm's fixed cost is $12,000 per month, determine the firm's break-even point.

Solution

The cost function C and the revenue function R are given by

$$C(x) = 4x + 12{,}000 \text{ and}$$

$$R(x) = 10x$$

Setting $R(x) = C(x)$, we get

$$10x = 4x + 12{,}000$$

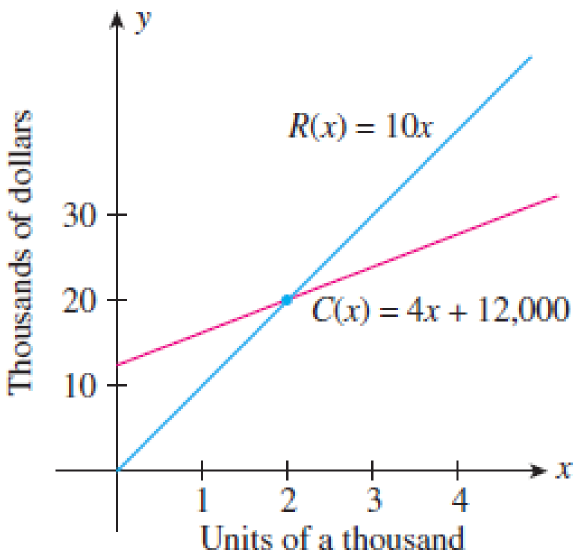
or $$x = 2000$$

Substituting this value of x into $R(x) = 10x$ gives

$$R(2000) = (10)(2000) = 20{,}000$$

So, for a break-even operation, the firm should manufacture 2000 units of its product, resulting in a break-even revenue of $20,000 per month.

The following figure illustrates the situation.



Exercise for Reader

1. On the Richter scale, the magnitude R of an earthquake is given by the formula $R = \log \frac{I}{I_o}$ where I is the intensity of the earthquake being measured and I_0 is the standard reference intensity.

   a) Express the intensity I of an earthquake of magnitude R = 5 in terms of the standard intensity I_0.

   b) Express the intensity I of an earthquake of magnitude R = 8 in terms of the standard intensity I_0. How many times greater is the intensity of an earthquake of magnitude 8 than one of magnitude 5?

   c) In modern times, the greatest loss of life attributable to an earthquake occurred in Nepal in 2015. Known as the Gorakha earthquake, it registered 7.8 on the Richter scale. How does the intensity of this earthquake compare with the intensity of an earthquake of magnitude R = 5?

2. Prescott manufactures its products at a cost of $4 per unit and sells them for $10 per unit. If the firm's fixed cost is $12,000 per month, determine the firm's break-even point.

   a) What is the loss sustained by the firm if only 1500 units are produced and sold each month?

   b) What is the profit if 3000 units are produced and sold each month?

   c) How many units should the firm produce in order to realize a minimum monthly profit of $9000?

3. A division of Carter Enterprises produces 'Personal Income Tax' diaries. Each diary sells for $8. The monthly fixed costs incurred by the division are $25,000, and the variable cost of producing each diary is $3.

   a) Find the break-even point for the division.

   b) What should be the level of sales in order for the division to realize a 15% profit over the cost of making the diaries?