Lecture 14

Learning Objectives

At the end of this class, students should be able to:

- evaluate the limit at infinity
- understand the concept of continuity
- solve related problems

14.1 Limits at Infinity

Up to now we have studied the limit of a function as x approaches a (finite) number c. There are occasions, however, when we want to know whether $f(x)$ approaches a unique number as x increases without bound.

Let us try to understand the concept with the help of the following calculation.

x	1	2	5	10	100	1000	10000	100000	1000000
$1/x$	1	0.5	0.2	0.1	0.01	0.001	0.0001	0.00001	0.000001

Here, we see that as x gets larger, $1/x$ tends towards 0.

We can't say what happens when x gets to infinity but we can see that as x gets larger and larger $1/x$ is going towards 0.

Thus, the limit of $1/x$ as x approaches infinity is 0. Mathematically,

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$$

It is a mathematical way of saying "we are not talking about when $x = \infty$, but we know as x gets bigger, the answer gets closer and closer to 0". In the general situation, the following definition for a limit of a function at infinity is applicable.

The function f has the limit L as x increases without bound (or, as x approaches infinity), written

$$\lim_{x \to \infty} f(x) = L$$

if $f(x)$ can be made very close to L by taking x large enough.

Similarly, the function f has the limit M as x decreases without bound (or as x approaches negative infinity), written

$$\lim_{x \to -\infty} f(x) = M$$

if $f(x)$ can be made very close to M by taking x to be negative and sufficiently large in absolute value.

Illustration 1

Evaluate $\lim_{x \to \infty} \left(\frac{5}{x+3} \right)$

Solution

$$\lim_{x \to \infty} \left(\frac{5}{x+3} \right) = \frac{5}{\infty + 3}$$

$$= \frac{5}{\infty}$$

$$= 0$$

Illustration 2

$\lim_{x \to \infty} \frac{2x+7}{x-1}$

Solution

$$\lim_{x \to \infty} \frac{2x+7}{x-1} = \lim_{x \to \infty} \frac{\frac{2x}{x} + \frac{7}{x}}{\frac{x}{x} - \frac{1}{x}}$$

[Numerator and denominator are divided by x]

$$= \lim_{x \to \infty} \frac{2 + \frac{7}{x}}{1 - \frac{1}{x}}$$

$$= \frac{2 + \frac{7}{\infty}}{1 - \frac{1}{\infty}}$$

$$= \frac{2+0}{1-0}$$

$$= 1$$

Illustration 3

Evaluate $\lim_{x \to \infty} \frac{5x}{3x^2 + 2x}$

Solution

$$\lim_{x \to \infty} \frac{5x}{3x^2 + 2x} = \lim_{x \to \infty} \frac{\frac{5x}{x^2}}{\frac{3x^2}{x^2} + \frac{2x}{x^2}}$$

[Numerator and denominator are divided by x^2]

$$= \lim_{x \to \infty} \frac{\frac{5}{x}}{3 + \frac{2}{x}}$$

$$= \frac{\frac{5}{\infty}}{3 + \frac{2}{\infty}}$$

$$= \frac{0}{3 + 0}$$

$= 0$

14.2 Continuity

Continuous functions play an important role in mathematics. Loosely speaking, a function is continuous at a point if the graph of the function at that point is devoid of holes, gaps, jumps, or breaks. Mathematically,

A function f is continuous at $x = c$ if the following conditions are satisfied

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists, and
3. $\lim_{x \to c} f(x) = f(c)$

Thus, a function f is continuous at $x = a$ if the limit of f at $x = a$ exists and has the value $f(a)$.

A function f is continuous over an interval $[a, b]$ if it is continuous at every point within the interval. It is said to be discontinuous at $x = c$ if it is not continuous at $x = c$.

Illustration 4

Check the continuity of the following function at $x = 0$.

$$f(x) = \begin{cases} -x + 1 & \text{for } x \le 0 \\ 2x + 3 & \text{for } x > 0 \end{cases}$$

Solution

Here, left hand limit at $x = 0$ is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x+1)$$

$$= - 0 + 1 \quad = 1$$

Right hand limit is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (2x+3)$$

$$= 2 \times 0 + 3 \ = 3$$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, therefore, $\lim_{x \to 0} f(x)$ does not exist.

Thus, the given function is not continuous at $x = 0$.

Illustration 5

Check the continuity of the following function at $x = 0$.

$$f(x) = \begin{cases} \dfrac{x-6}{x-3} & \text{for } x < 0 \\ 2 & \text{for } x = 0 \\ \sqrt{4+x^2} & \text{for } x > 0 \end{cases}$$

Solution

Here, left hand limit is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{x-6}{x-3}$$

$$= \frac{0-6}{0-3} = 2$$

Right hand limit is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \sqrt{4+x^2}$$

$$= \sqrt{4+0^2} = 2$$

The value of the function at $x = 0$ is

$$f(0) = 2$$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ therefore the given function is continuous at $x = 0$.

Continuity of Polynomial and Rational Functions

1. A polynomial function $f(x)$ is continuous at every value of *x*.

2. A rational function $f(x) = \dfrac{g(x)}{h(x)}$ is continuous at every value of *x* where $h(x) \neq 0$

Exercise for Reader

1. Evaluate the following limits.

   a) $\lim\limits_{x \to \infty} \dfrac{x-5}{3x+8}$

   b) $\lim\limits_{x \to \infty} \dfrac{2x^3+8x+10}{x^3-x}$

2. Check the continuity of the following function at *x* = 0.

$$f(x) = \begin{cases} 2x-4 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

3. Check the continuity of the following function at *x* = 0.

$$f(x) = \begin{cases} x^2+1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

4. Check the continuity of the following function at *x* = 1.

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$$

5. Check the continuity of the following function at *x* = 0.

$$f(x) = \begin{cases} x+5 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ -x^2+5 & \text{if } x > 0 \end{cases}$$