Lecture 16

Learning Objectives

At the end of this class, students should be able to:

- use differentiation rules

Differentiation Rules

1. **Product Rule:** If $f(x) = u(x).v(x)$, where u and v are differentiable functions, then $f'(x) = u(x)\,v'(x) + v(x)u'(x)$.

Illustration 1

Let $f(x) = (3x^3 + 5x)(4 - 7x)$, then find $f'(x)$.

Solution

We have $f(x) = (3x^3 + 5x)(4 - 7x)$, then

$$\begin{aligned} f'(x) &= (3x^3 + 5x)\frac{d}{dx}(4 - 7x) + (4 - 7x)\frac{d}{dx}(3x^3 + 5x) \\ &= (3x^3 + 5x)(-7) + (4 - 7x)(9x^2 + 5) \\ &= -21x^3 - 35x + 36x^2 + 20 - 63x^3 - 35x \\ &= -84x^3 - 70x + 36x^2 + 20 \end{aligned}$$

2. **Quotient Rule:** If $f(x) = u(x)/v(x)$, where u and v are differentiable functions and $v(x) \neq 0$, then

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Illustration 2

Let $f(x) = \dfrac{2x}{5 - 3x}$, then find $f'(x)$.

Solution

We have $f(x) = \dfrac{2x}{5 - 3x}$, then

$$f'(x) = \frac{(5-3x)\times\frac{d}{dx}(2x) - 2x\times\frac{d}{dx}(5-3x)}{(5-3x)^2}$$

$$= \frac{(5-3x)\times 2 - 2x\times(-3)}{(5-3x)^2}$$

$$= \frac{10-6x+6x}{(5-3x)^2}$$

$$= \frac{10}{(5-3x)^2}$$

3. General Power Rule: If $f(x) = [u(x)]^n$, where *u* is a function of *x* and *n* is a real number, then $f'(x) = n[u(x)]^{n-1}.\ u'(x)$

Illustration 3

Let $f(x) = (5-x^3)^{1/2}$, then find *f* '(*x*).

Solution

We have $f(x) = (5-x^3)^{1/2}$, then

$$f'(x) = \frac{d}{dx}(5-x^3)^{1/2}$$

$$= \frac{1}{2}(5-x^3)^{-1/2}\frac{d}{dx}(5-x^3)$$

$$= \frac{1}{2\sqrt{(5-x^3)}}(-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{5-x^3}}$$

4. Chain Rule: If *z* = *f* (*y*), and *y* = *g* (*x*), then the derivative of *z* with respect to *x* is given by the formula: $\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx}$

Illustration 4

Let $z = 3y^2 - 5y$ and $y = x^2 - 3$, then find $\frac{dz}{dx}$.

Solution

We have $z = 3y^2 - 5y$ and $y = x^2 - 3$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$= \frac{d}{dy}(3y^2 - 5y) \times \frac{d}{dx}(x^2 - 3)$$

$$= (6y - 5) \times 2x$$

$$= 2x\{6(x^2 - 3) - 5\}$$

$$= 12x^3 - 46x$$

5. Implicit Differentiation: Sometimes functions are given not in the form $y = f(x)$ but in a more complicated form in which it is difficult or impossible to express y explicitly in terms of x. Such functions are called implicit functions. Implicit differentiation is a special case of the chain rule for derivatives. Let us try to understand the process with the help of the following illustration.

Illustration 5

Let $10x^4 - 18xy^2 + 10y^3 = 48$ then find $\frac{dy}{dx}$.

Solution

We have $10x^4 - 18xy^2 + 10y^3 = 48$, then differentiating both sides with respect to x, we get

$$10\,\frac{d}{dx}(x^4) - 18\,\frac{d}{dx}(xy^2) + 10\frac{d}{dx}(y^3) = \frac{d}{dx}(48)$$

or, $$40x^3 - 18\left\{x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)\right\} + 10\frac{d}{dy}(y^3)\frac{dy}{dx} = 0$$

$$\left[\because \frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}\right]$$

or, $$40x^3 - 18\left\{x\frac{d}{dy}(y^2)\frac{dy}{dx} + y^2 \times 1\right\} + 10 \times 3y^2\frac{dy}{dx} = 0$$

or, $$40x^3 - 18\left\{x \times 2y\frac{dy}{dx} + y^2\right\} + 30y^2\frac{dy}{dx} = 0$$

or, $$40x^3 - 36xy\frac{dy}{dx} - 18y^2 + 30y^2\frac{dy}{dx} = 0$$

or, $$(30y^2 - 36xy)\frac{dy}{dx} = 18y^2 - 40x^3$$

or, $$\frac{dy}{dx} = \frac{18y^2 - 40x^3}{30y^2 - 36xy}$$

$\therefore$ $$\frac{dy}{dx} = \frac{9y^2 - 20x^3}{15y^2 - 18xy}$$

Exercise for Reader

1. Find the derivative of the following functions with respect to *x*.
   a) $f(x) = (4-3x)(x^2 - x + 2)$
   b) $f(x) = \dfrac{x+1}{2x-1}$
   c) $f(x) = \dfrac{5x^2 - 7x}{3x^2 + 2}$
   d) $f(x) = (3-2x)^5$
   e) $f(x) = (2x^2 + x)^3$
2. If $y = 5u^2 - 7u$ and $u = x^2 + 3$ then find the value of $\dfrac{dy}{dx}$.
3. If $y = x^2y^3 + x^3y^2$ then find the value of $\dfrac{dy}{dx}$.