Lecture 17

Learning Objectives

At the end of this class, students should be able to:

- use differentiation rules
- identify higher order derivatives
- identify rate of change

17.1 Differentiation Rules

1. Derivative of Exponential Function: Let $f(x) = e^x$ then $f'(x) = e^x$ and if $f(x) = e^{u(x)}$ then $f'(x) = e^{u(x)} \times \frac{d}{dx}(u(x))$, where *u* is a function of *x*.

Illustration 1

Let $f(x) = e^{-3x^2+4x}$, then find $f'(x)$.

Solution

We have $f(x) = e^{-3x^2+4x}$, then

$$f'(x) = e^{-3x^2+4x} \times \frac{d}{dx}(-3x^2+4x)$$

$$= (-6x+4)\, e^{-3x^2+4x}$$

Note: If $f(x) = a^x$ then $f'(x) = a^x \ln a$

2. Derivative of Logarithmic Function:

Let $f(x) = \ln x$ then $f'(x) = \frac{1}{x}, \quad x > 0$ and

If $f(x) = \ln u(x)$ then $f'(x) = \frac{1}{u(x)}\left[\frac{d}{dx}\{u(x)\}\right]$

Illustration 2

Let $f(x) = \ln(3x^2+7)$, then find $f'(x)$.

Solution

We have $f(x) = \ln(3x^2+7)$, then

$$f'(x) = \frac{d}{dx}\{\ln(3x^2+7)\}$$

$$= \frac{1}{3x^2+7} \times \frac{d}{dx}(3x^2+7)$$

$$= \frac{6x}{3x^2+7}$$

Note: If $f(x) = \log_a x$ then $f'(x) = \frac{1}{x(\ln a)}, \; x > 0$

17.2 Higher Order Derivatives

Consider the function given by

$$y = f(x) = x^5 + x^3 - x^2 + 15$$

Its first order derivative is given by

$$\frac{dy}{dx} = f'(x) = 5x^4 + 3x^2 - 2x$$

The derivative function $f'(x)$ can also be differentiated. We use the notation $f''(x)$ for the derivative of $f'(x)$.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x) = \frac{d}{dx}(5x^4 + 3x^2 - 2x)$$

or $$\frac{d^2y}{dx^2} = f''(x) = 20x^3 + 6x - 2$$

Continuing in the same manner, we are led to considering the third, fourth, and higher-order derivatives of f whenever they exist.

$$\frac{d^3y}{dx^3} = f'''(x) = 60x^2 + 6,$$

$$\frac{d^4y}{dx^4} = f''''(x) = 120x,$$

$$\frac{d^5y}{dx^5} = f'''''(x) = 120$$

When notations like $f''''(x)$ gets lengthy, we abbreviate it using a number or n in parentheses.

Thus, $f^{(n)}(x)$ is the nth derivative. For the function above,

$$f^{(4)}(x) = 120x;$$

$$f^{(5)}(x) = 120;$$

and, in general,

$$f^{n}(x) = 0 \quad (n > 5)$$

Illustration 3

Find $f'(x)$ and $f''(x)$ if $f(x) = (x^2 + 5x)^{10}$.

Solution

Here, $f(x) = (x^2 + 5x)^{10}$ then

$$f'(x) = 10(x^2 + 5x)^9 \frac{d}{dx}(x^2 + 5x)$$

$$= 10(x^2 + 5x)^9 (2x + 5)$$

Again,

$$f''(x) = 10\left[(x^2 + 5x)^9 \frac{d}{dx}(2x + 5) + (2x + 5)\frac{d}{dx}(x^2 + 5x)^9\right]$$

$$= 10\left[(x^2 + 5x)^9 \times 2 + (2x + 5) \times 9(x^2 + 5x)^8 \times \frac{d}{dx}(x^2 + 5x)\right]$$

$$= 10\left[2(x^2 + 5x)^9 + (2x + 5) \times 9(x^2 + 5x)^8 \times (2x + 5)\right]$$

$$= 10(x^2 + 5x)^8 \left[2(x^2 + 5x) + 9(2x + 5)^2\right]$$

$$= 10(x^2 + 5x)^8 (2x^2 + 10x + 36x^2 + 180x + 225)$$

$$= 10(x^2 + 5x)^8 (38x^2 + 190x + 225)$$

17.3 The Derivative as a Rate of Change

Since the derivative of a function $f(x)$ measures the rate of change of the function with respect to *x*. Thus, the derivative is used to determine instantaneous rate of change.

Illustration 4

A developing country's gross domestic product (GDP) from 2000 to 2008 is approximated by the function $G(t) = -0.2t^3 + 2.4t^2 + 60; (0 \le t \le 8)$ where $G(t)$ is measured in billions of dollars, with $t = 0$ corresponding to the beginning of 2000. Compute *G'*(5) and interpret your result.

Solution

Here $G'(t) = -0.6t^2 + 4.8t$, then

$$G'(5) = -0.6 \times 5^2 + 4.8 \times 5$$

$$= 9$$

That is, the GDP is increasing at the rate of \$9 billion per year at the end of 2005.

Illustration 5

The sales (in millions of dollars) of a DVD recording of a hit movie *t* years from the date of release is given by $S(t) = \dfrac{5t}{t^2 + 1}$

a) Find the rate at which the sales are changing at time *t*.

b) How fast are the sales changing at the time the DVDs are released ($t = 0$)? Two years from the date of release?

Solution

a) The rate at which the sales are changing at time *t* is given by *S'*(*t*). Using the quotient rule, we obtain

$$S'(t) = \frac{(t^2+1)\frac{d}{dt}(5t) - 5t\frac{d}{dt}(t^2+1)}{(t^2+1)^2}$$

$$= \frac{(t^2+1)\times 5 - 5t \times 2t}{(t^2+1)^2}$$

$$= \frac{-5t^2+5}{(t^2+1)^2}$$

b) The rate at which the sales are changing at the time the DVDs are released is given by

$$S'(0) = \frac{-5\times 0+5}{(0+1)^2}$$

$$= 5$$

That is, the sales of a DVD recording are increasing at the rate of $5 million per year.

Two years from the date of release, the sales are changing at the rate of

$$S'(2) = \frac{-5\times 2^2+5}{(2^2+1)^2}$$

$$= -0.6$$

That is, the sales of a DVD recording are decreasing at the rate of $600,000 per year.

Exercise for Reader

1. Find the derivative of the following functions with respect to *x*.

   a) $f(x) = xe^{2x}$

   b) $f(x) = \dfrac{e^{2x}}{1+e^{-2x}}$

   c) $f(x) = \dfrac{\ln x}{x+1}$

   d) $f(x) = \dfrac{e^{x}}{1+\ln x}$

2. If $e^{xy} = xy^3$ then find the value of $\dfrac{dy}{dx}$.

3. The demand function for the Sicard wristwatch is given by $D(x) = \dfrac{50}{0.01x^2+1}$; $(0 \le x \le 20)$ where *x* (measured in units of a thousand) is the quantity demanded per week and *D*(*x*) is the unit price in dollars.

   a) Find *D'*(*x*).

   b) Find *D'*(15) and interpret your results.

4. The number of viewers of a television series introduced several years ago is approximated by the function $N(t) = (60+2t)^{2/3}$; $(1 \le t \le 26)$ where *N*(*t*) (measured in millions) denotes the number of weekly viewers of the series in the t^{th} week. Find the rate of increase of the weekly audience at the end of week 2 and at the end of week 12. How many viewers were there in week 2? In week 24?

5. The registrar of Kellogg University estimates that the total student enrollment in the Continuing Education division will be given by $N(t) = -\dfrac{20{,}000}{\sqrt{1+0.2t}} + 21{,}000$ where *N*(*t*) denotes the number of students enrolled in the division *t* year from now. Find an expression for *N'*(*t*). How fast is the student enrollment increasing currently? How fast will it be increasing 5 years from now?