Lecture 18

Learning Objectives

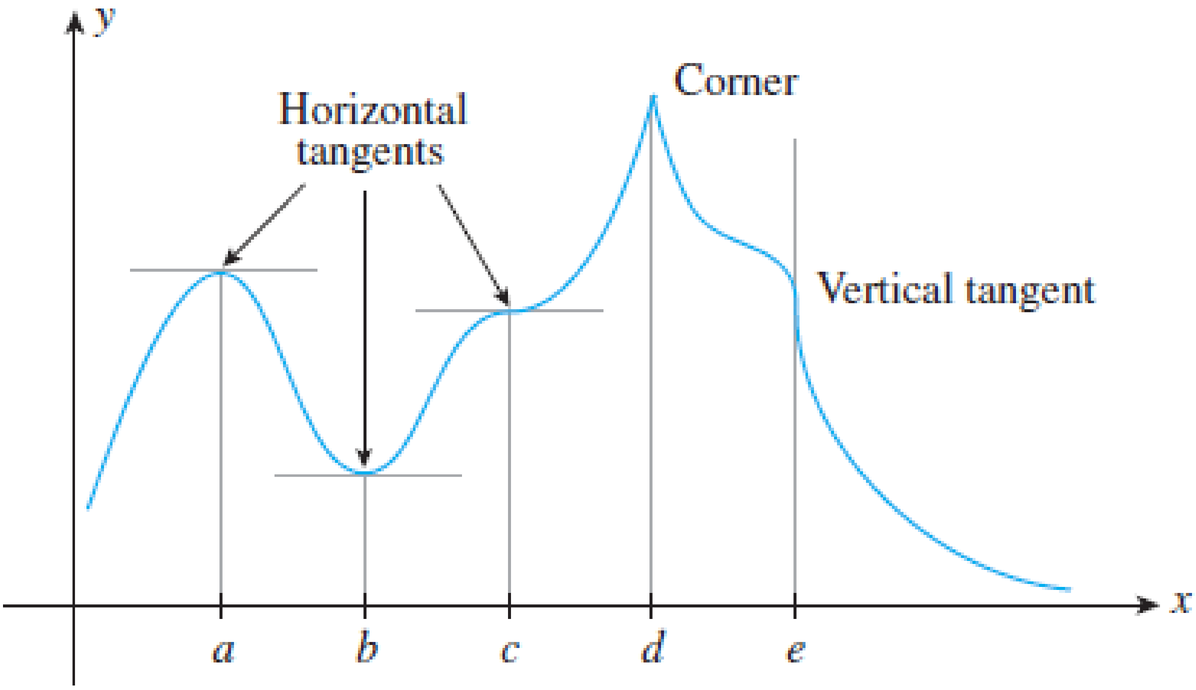
At the end of this class, students should be able to:

- identify relative maximum and minimum

18.1 Critical Values

A critical value of a function *f* is any number *x* in the domain of *f* such that $f'(x) = 0$ or $f'(x)$ does not exist.

Let us consider the following graph of a function $f(x)$.



In the above figure, the graph of a function has critical values at *a*, *b*, *c*, *d*, and *e*. We observe that $f'(x) = 0$ at *a*, *b*, and *c*. $f'(x)$ is not defined at *d* because the function has a corner at that point. Similarly, $f'(x)$ does not exist at *e* because the tangent line is vertical at that point.

A 'peak' on the graph of a function *f* is known as a relative maximum of *f*, and a 'valley' is a relative minimum. Thus, a relative maximum is a point on the graph of *f* that is at least as high as any nearby point on the graph, while a relative minimum is at least as low as any nearby point.

In the above figure, relative maximum occurs at *a* and *d* whereas relative minimum occurs at *b*. We have to apply standard test to identify whether the function has relative maximum or minimum at the particular critical point.

18.2 The Second Derivative Test

The following steps will be followed for identifying optimum values of the function.

1. Find all critical points x^*, such that $f'(x^*)$ = 0.
2. For any critical point x^*, determine the value of $f''(x^*)$.
   a) If $f''(x^*)$ > 0, the given function has a relative minimum at x^*.
   b) If $f''(x^*)$ < 0, the given function has a relative maximum at x^*.
   c) If $f''(x^*)$ = 0, no conclusion can be drawn. Another test is required.

Illustration 1

Find the local maxima and minima for function $f(x) = x^3 - 6x^2 + 9x + 5$.

Solution

We have $f(x) = x^3 - 6x^2 + 9x + 5$ then

$$f'(x) = 3x^2 - 12x + 9$$

For critical value, setting $f'(x) = 0$

i.e. $3x^2 - 12x + 9 = 0$

or $3(x-1)(x-3) = 0$

Thus, the critical values are $x = 1$ and $x = 3$.

Now, $f''(x) = 6x - 12$

When x = 1,

$$f''(1) = 6 \times 1 - 12 = -6 < 0$$

Thus, the given function has a relative maximum at x = 1.

When x = 3,

$$f''(3) = 6 \times 3 - 12 = 6 > 0$$

Thus, the given function has a relative minimum at x = 3.

Illustration 2

A company manufactures and sells x videophones per week. The weekly price demand and cost equations are, respectively, $p = 500 - 0.5x$ and $C(x) = 20{,}000 + 135x$

a) What price should the company charge for the phones, and how many phones should be produced to maximize the weekly revenue? What is the maximum weekly revenue?
b) What is the maximum weekly profit? How much should the company charge for the phones, and how many phones should be produced to realize the maximum weekly profit?

Solution

a) We know that,

Revenue = price × demand

i.e. $R = (500 - 0.5x)x$

or $R = 500x - 0.5x^2$

Both price and demand must be nonnegative, so

$x \geq 0$ and $p = 500 - 0.5x \geq 0$

i.e. $500 \geq 0.5x$

or $x \leq 1000$

Thus, the mathematical model for this problem is

$$R = 500x - 0.5x^2, \qquad 0 \leq x \leq 1000$$

Here, $R' = 500 - x$

For maximum revenue, set $R' = 0$

i.e. $500 - x = 0 \quad \Rightarrow x = 500$

Again,

$R'' = -1$

When x = 500

$R'' = -1 < 0$

Thus, the revenue is maximum when 500 phones are produced and sold.

The corresponding price for maximum revenue is

$$p = 500 - 0.5 \times 500 = \$250 .$$

Weekly maximum revenue is $R = \$250 \times 500 = \$12{,}500$.

b) We have $R = 500x - 0.5x^2$ and $C(x) = 20{,}000 + 135x$

Here, profit $P = R - C = 500x - 0.5x^2 - 20{,}000 - 135x$

or $P = -0.5x^2 + 365x - 20{,}000$

Now, $P' = -x + 365$

For maximum profit, set $P' = 0$

i.e. $-x + 365 = 0 \quad \Rightarrow x = 365$

Again,

$P'' = -1$

When x = 365

$P'' = -1 < 0$

Thus, the profit is maximum when 365 phones are produced and sold.

The corresponding price for maximum profit is

$$p = 500 - 0.5 \times 365 = \$317.50.$$

Weekly maximum profit is $P = -0.5 \times (365)^2 + 365 \times 365 - 20{,}000$

= \$46,612.50.

Exercise for Reader

1. Find the relative extrema of the following functions.

   a) $f(x) = 2x^2 + 3x - 4$

   b) $f(x) = x^3 + 3x^2 - 1$

   c) $f(x) = x^3 - 2x^2 - 4x + 4$

2. A manufacturer of tennis rackets finds that the total cost $C(x)$ (in dollars) of manufacturing x rackets per day is given by $C(x) = 400 + 4x + 0.0001x^2$. Each racket can be sold at a price of p dollars, where p is related to x by the demand equation $p = 10 - 0.0004x$. If all rackets that are manufactured can be sold, find the daily level of production that will yield a maximum profit for the manufacturer.

3. The management of Trappee and Sons, producers of the famous TexaPep hot sauce, estimate that their profit (in dollars) from the daily production and sale of x cases (each case consisting of 24 bottles) of the hot sauce is given by $P(x) = -0.000002x^3 + 6x - 400$ What is the largest possible profit Trappee can make in 1 day?