**Lecture 20**

**Learning Objectives**

At the end of this class, students should be able to:

- identify absolute maximum and minimum
- solve the problems related to elasticity of demand

## 20.1 Functions of Several Variables

An equation of the form $u = f(x, y)$ describes a function of two independent variables if, for each permissible ordered pair (*x*, *y*), there is one and only one value of *u* determined by $f(x, y)$. The variables *x* and *y* are independent variables, and the variable *u* is a dependent variable.

The set of all ordered pairs of permissible values of *x* and *y* is the domain of the function, and the set of all corresponding values $f(x, y)$ is the range of the function. Similarly, $u = f(x, y, z)$ is the function of three variables.

***Illustration 1***

Let *f* be the function defined by $f(x, y) = x^2 + xy + 5$. Compute *f* (0, 0) and *f* (1, 3).

***Solution***

Here $\quad f(0,0) = 0^2 + 0 \times 0 + 5 = 5$

and $\quad f(1,3) = 1^2 + 1 \times 3 + 5 = 9$

***Illustration 2***

Pradhan Workshop manufactures both finished and unfinished furniture for the home. The estimated quantities demanded each week of its rolltop desks in the finished and unfinished versions are *x* and *y* units when the corresponding unit prices are

$$p = 200 - \frac{1}{5}x - \frac{1}{10}y$$

$$q = 160 - \frac{1}{10}x - \frac{1}{4}y$$

dollars, respectively. What is the weekly total revenue function *R*(*x*, *y*)?

***Solution***

We know that

Revenue = Price × Quantity

Revenue generated from finished furniture = $(200-\frac{1}{5}x-\frac{1}{10}y)x$

Revenue generated from unfinished furniture = $(160-\frac{1}{10}x-\frac{1}{4}y)y$

The weekly total revenue is given by

$$R(x,y) = (200-\frac{1}{5}x-\frac{1}{10}y)x + (160-\frac{1}{10}x-\frac{1}{4}y)y$$

$$= 200x-\frac{1}{5}x^2-\frac{1}{10}xy+160y-\frac{1}{10}xy-\frac{1}{4}y^2$$

$$= -\frac{1}{5}x^2-\frac{1}{4}y^2-\frac{1}{5}xy+200x+160y$$

**20.2 Level Curves**

Let $z = f(x,\ y)$ be a function of two independent variables. A set of points at which $f$ has the same value is called a *level curve*.

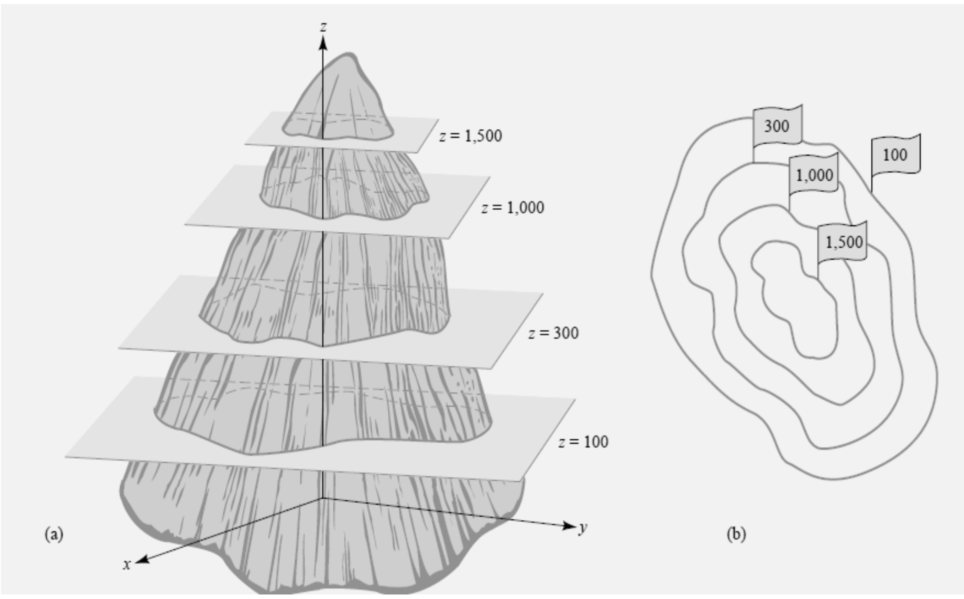
So, a level curve for a function $z = f(x,\ y)$ is a curve in the *xy*-plane defined by $f(x,\ y) = c$, where c is a constant. As c varies, we get a family of level curves.

A concept of level curves has many interesting applications.

a) Surveyors draw topographic maps in which level curves are used to represent points having same altitude. They are curves of constant elevation above sea level. If we move along one of the level curves, we neither ascend nor descend. We are at the same height.

b) The level curves of temperature function is another example of such curves which are called isothermals.

In the following figure (a), the surface $z = f(x, y)$ represents the mountain and figure (b) shows the level curves which is a topographical map of the mountain.



## 20.3 Partial Derivatives

Let $u = f(x, y)$ then we find the partial derivative of *u* with respect to *x* (denoted by $\frac{\partial u}{\partial x}$) by treating the variable *y* as a constant and taking the derivative of $u = f(x, y)$ with respect to *x*. The other notations to represent partial derivative of $u = f(x, y)$ with respect to *x* are $\frac{\partial f}{\partial x}$, $\frac{\partial}{\partial x} f(x, y)$, $f_x$, and $u_x$.

Similarly, we can also take the partial derivative of *u* with respect to *y* by holding the variable *x* constant and taking the derivative of $u = f(x, y)$ with respect to *y*. We denote this derivative as $\frac{\partial u}{\partial y}$.

Note that $\frac{du}{dx}$ represents the derivative of a function of one variable $u = f(x)$ whereas, $\frac{\partial u}{\partial x}$ represents the partial derivative of a function of two or more variables.

***Illustration 3***

Find the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ of the function $u = 5x^2 y^3$.

***Solution***

We have $u = 5x^2 y^3$ then

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(5x^2 y^3)$$

$$= 5y^3 \frac{\partial}{\partial x}(x^2)$$

$$= 5y^3 \times 2x$$

$$= 10xy^3$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(5y^3 x^2)$$

$$= 5x^2 \frac{\partial}{\partial y}(y^3)$$

$$= 5x^2 \times 3y^2$$

$$= 15x^2 y^2$$

**Exercise for Reader**

1. Weston Publishing publishes a deluxe edition and a standard edition of its English language dictionary. Weston's management estimates that the number of deluxe editions demanded is *x* copies/day and the number of standard editions demanded is *y* copies/day when the unit prices are

   $$p = 20 - 0.005x - 0.001y$$

   $$p = 15 - 0.001x - 0.003y$$

   dollars, respectively. Find the daily total revenue function $R(x, y)$.

2. Suppose the output of a certain country is given by $f(x, y) = 100x^{3/5}y^{2/5}$ billion dollars if *x* billion dollars are spent for labor and *y* billion dollars are spent on capital. Find the output if the country spent $32 billion on labor and $243 billion on capital.

3. Find the first order partial derivatives of each function.

   a) $f(x, y) = 2x^2 y$

   b) $f(x, y) = 100x^{3/5}y^{2/5}$