Lecture 21

Learning Objectives

At the end of this class, students should be able to:

- apply differential rules

The rules of partial differentiation follow exactly the same logic as simple differentiation. Here, we have to decide how to treat the remaining variable.

21.1 Sum or Difference Rule

If $u = f(x, y) = g(x, y) \pm h(x, y)$, where g and h are differentiable functions, then

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial x} \pm \frac{\partial h}{\partial x} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = \frac{\partial g}{\partial y} \pm \frac{\partial h}{\partial y}$$

Let us try to understand this rule with the help of the following illustration.

Illustration 1

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function: $f(x, y) = x^2 + xy^2 - y^3$.

Solution

We have $f(x, y) = x^2 + xy^2 - y^3$ then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy^2 - y^3)$$

$$= 2x + y^2 - 0$$

$$= 2x + y^2$$

Again

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy^2 - y^3)$$

$$= 0 + 2xy - 3y^2$$

$$= 2xy - 3y^2$$

21.2 Product Rule

If $u = f(x, y) = g(x, y).h(x, y)$, where g and h are differentiable functions, then

$$\frac{\partial u}{\partial x} = g\frac{\partial h}{\partial x} + h\frac{\partial g}{\partial x} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = g\frac{\partial h}{\partial y} + h\frac{\partial g}{\partial y}$$

Let us try to understand this rule with the help of the following illustration.

Illustration 2

Find the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ of the function: $u = (3x + 2y)(5 - 2x)$.

Solution

We have $u = (3x + 2y)(5 - 2x)$ then

$$\frac{\partial u}{\partial x} = (3x + 2y)\frac{\partial}{\partial x}(5 - 2x) + (5 - 2x)\frac{\partial}{\partial x}(3x + 2y)$$

$$= (3x + 2y)(-2) + (5 - 2x)(3)$$

$$= -12x - 4y + 15$$

$$\frac{\partial u}{\partial y} = (3x + 2y)\frac{\partial}{\partial y}(5 - 2x) + (5 - 2x)\frac{\partial}{\partial y}(3x + 2y)$$

$$= (3x + 2y)(0) + (5 - 2x)(2)$$

$$= -4x + 10$$

21.3 Quotient Rule

If $u = f(x, y) = g(x, y) / h(x, y)$ *and* $h(x, y) \neq 0$, where g and h are differentiable functions, then

$$\frac{\partial u}{\partial x} = \frac{h\frac{\partial g}{\partial x} - g\frac{\partial h}{\partial x}}{h^2} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = \frac{h\frac{\partial g}{\partial y} - g\frac{\partial h}{\partial y}}{h^2}$$

Let us try to understand this rule with the help of the following illustration.

Illustration 3

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $f(x,\ y) = \frac{5y}{x^2 + y^2}$.

Solution

We have $f(x,\ y) = \frac{5y}{x^2 + y^2}$ then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{5y}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2) \times \frac{\partial}{\partial x}(5y) - 5y \times \frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2) \times 0 - 5y \times 2x}{(x^2 + y^2)^2}$$

$$= \frac{-10xy}{(x^2 + y^2)^2}$$

Again,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{5y}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2) \times \frac{\partial}{\partial y}(5y) - 5y \times \frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2) \times 5 - 5y \times 2y}{(x^2 + y^2)^2}$$

$$= \frac{5x^2 + 5y^2 - 10y^2}{(x^2 + y^2)^2}$$

$$= \frac{5(x^2 - y^2)}{(x^2 + y^2)^2}$$

Exercise for Reader

Find the first order partial derivatives of each function.

a) $f(x, y) = 2x^2 + 4y + 5$

b) $f(x, y) = 3x^2 y^4 + y^2$

c) $f(x, y) = (x + y)(2x^3 - 5y)$

d) $f(x, y) = \dfrac{x}{1 + y}$

e) $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$