

Lecture 22

Learning Objectives

At the end of this class, students should be able to:

- apply differential rules
- identify the marginal values

22.1 Generalized Power Rule

If $u = [f(x, y)]^n$, then

$$\frac{\partial u}{\partial x} = n[f(x, y)]^{n-1} \cdot \frac{\partial f}{\partial x} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = n[f(x, y)]^{n-1} \cdot \frac{\partial f}{\partial y}$$

Let us try to understand this rule with the help of the following illustration.

Illustration 1

Find the partial derivatives f_x and f_y of the function $f(x, y) = \sqrt{x^2 + y^2}$.

Solution

We have $f(x, y) = \sqrt{x^2 + y^2}$ then

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} (x^2 + y^2)^{1/2} \\ &= \frac{1}{2} (x^2 + y^2)^{-1/2} \frac{\partial}{\partial x} (x^2 + y^2) \\ &= \frac{1}{2\sqrt{(x^2 + y^2)}} \times 2x \\ &= \frac{x}{\sqrt{(x^2 + y^2)}} \text{ convenience} \end{aligned}$$

Again,

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} (x^2 + y^2)^{1/2} \\ &= \frac{1}{2} (x^2 + y^2)^{-1/2} \frac{\partial}{\partial y} (x^2 + y^2) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{(x^2 + y^2)}} \times 2y \\
&= \frac{y}{\sqrt{(x^2 + y^2)}}
\end{aligned}$$

Illustration 2

Find the partial derivatives f_x and f_y of the function $f(x, y) = \ln(x^2 y + 1) - ye^{-x}$.

Solution

We have $f(x, y) = \ln(x^2 y + 1) - ye^{-x}$ then

$$\begin{aligned}
f_x &= \frac{\partial}{\partial x} (\ln(x^2 y + 1) - ye^{-x}) \\
&= \frac{\partial}{\partial x} \ln(x^2 y + 1) - y \frac{\partial}{\partial x} (e^{-x}) \\
&= \frac{1}{(x^2 y + 1)} \frac{\partial}{\partial x} (x^2 y + 1) - ye^{-x} \frac{\partial}{\partial x} (-x) \\
&= \frac{1}{(x^2 y + 1)} \times 2xy - ye^{-x} \times (-1) \\
&= \frac{2xy}{(x^2 y + 1)} + ye^{-x}
\end{aligned}$$

Again,

$$\begin{aligned}
f_y &= \frac{\partial}{\partial y} (\ln(x^2 y + 1) - ye^{-x}) \\
&= \frac{\partial}{\partial y} \ln(x^2 y + 1) - e^{-x} \frac{\partial}{\partial y} (y) \\
&= \frac{1}{(x^2 y + 1)} \frac{\partial}{\partial y} (x^2 y + 1) - e^{-x} \times 1 \\
&= \frac{1}{(x^2 y + 1)} \times x^2 - e^{-x} \\
&= \frac{x^2}{(x^2 y + 1)} - e^{-x}
\end{aligned}$$

Illustration 3

Find the partial derivatives f_x , f_y and f_z of the function: $f(x, y, z) = 2xyz + xy^2 + yz^2 + zx^2$.

Solution

We have $f(x, y, z) = 2xyz + xy^2 + yz^2 + zx^2$ then

$$\begin{aligned}f_x &= \frac{\partial}{\partial x}(2xyz + xy^2 + yz^2 + zx^2) \\&= 2yz \frac{\partial}{\partial x}(x) + y^2 \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(yz^2) + z \frac{\partial}{\partial x}(x^2) \\&= 2yz \times 1 + y^2 \times 1 + 0 + z \times 2x \\&= 2yz + y^2 + 2xz\end{aligned}$$

Again,

$$\begin{aligned}f_y &= \frac{\partial}{\partial y}(2xyz + xy^2 + yz^2 + zx^2) \\&= 2xz \frac{\partial}{\partial y}(y) + x \frac{\partial}{\partial y}(y^2) + z^2 \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}(zx^2) \\&= 2xz \times 1 + x \times 2y + z^2 \times 1 + 0 \\&= 2xz + 2xy + z^2\end{aligned}$$

Similarly,

$$\begin{aligned}f_z &= \frac{\partial}{\partial z}(2xyz + xy^2 + yz^2 + zx^2) \\&= 2xy \frac{\partial}{\partial z}(z) + \frac{\partial}{\partial z}(xy^2) + y \frac{\partial}{\partial z}(z^2) + x^2 \frac{\partial}{\partial z}(z) \\&= 2xy \times 1 + 0 + y \times 2z + x^2 \times 1 \\&= 2xy + 2yz + x^2\end{aligned}$$

Marginal Analysis

The marginal analysis refers to the practice of using a derivative to estimate the change in the value of a function resulting from a 1-unit increase in one of its variables. The following examples are related to marginal analysis.

Illustration 4

Lincolntonville Sporting Goods has the production function $p(x, y) = 2400x^{2/5}y^{3/5}$ for a certain product: where p is the number of units produced with x units of labor and y units of capital.

- Find the number of units produced with 32 units of labor and 1024 units of capital.
- Find the marginal productivities.
- Evaluate the marginal productivities at $x = 32$ and $y = 1024$.
- Interpret the meanings of the marginal productivities found in part (c).

Solution

We have $p(x, y) = 2400x^{2/5}y^{3/5}$

$$\begin{aligned} \text{a) } p(32, 1024) &= 2400 \times (32)^{2/5} \times (1024)^{3/5} \\ &= 2400 \times 4 \times 64 \\ &= 614,400 \end{aligned}$$

b) The marginal productivity of labor is:

$$\frac{\partial p}{\partial x} = 2400y^{3/5} \frac{\partial}{\partial x}(x^{2/5}) = 960 \frac{y^{3/5}}{x^{3/5}}$$

The marginal productivity of capital is:

$$\frac{\partial p}{\partial y} = 2400x^{2/5} \frac{\partial}{\partial y}(y^{3/5}) = 1440 \frac{x^{2/5}}{y^{2/5}}$$

c) When $x = 32$ and $y = 1024$,

$$\begin{aligned} \text{Marginal productivity of labor} &= \left. \frac{\partial p}{\partial x} \right|_{(32, 1024)} \\ &= 960 \frac{(1024)^{3/5}}{(32)^{3/5}} = \frac{960 \times 64}{8} = 7,680 \end{aligned}$$

$$\begin{aligned} \text{Marginal productivity of capital} &= \left. \frac{\partial p}{\partial y} \right|_{(32, 1024)} \\ &= 1440 \frac{(32)^{2/5}}{(1024)^{2/5}} = \frac{1440 \times 4}{16} = 360 \end{aligned}$$

d) Suppose that capital is fixed at 1024 units. Then the marginal productivity of labor tells us that a one-unit change in labor, from 32 to 33, will cause production to increase by about 7,680 units.

Similarly, suppose that labor is fixed at 32 units. Then the marginal productivity of capital tells us that a one-unit change in capital, from 1024 to 1025, will cause production to increase by about 360 units

Illustration 5

The total weekly revenue (in dollars) of Shrestha Workshop associated with manufacturing and selling their rolltop desks is given by the function

$$R(x, y) = -0.2x^2 - 0.25y^2 - 0.2xy + 200x + 160y$$

where x denotes the number of finished units and y denotes the number of unfinished units manufactured and sold each week.

Compute $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$ when $x = 300$ and $y = 250$. Interpret your results.

Solution

We have $R(x, y) = -0.2x^2 - 0.25y^2 - 0.2xy + 200x + 160y$ then

$$\frac{\partial R}{\partial x} = -0.4x - 0.2y + 200 \text{ and}$$

$$\frac{\partial R}{\partial y} = -0.5y - 0.2x + 160$$

When $x = 300$ and $y = 250$,

$$\frac{\partial R}{\partial x} = -0.4 \times 300 - 0.2 \times 250 + 200 = 30$$

Thus, the weekly revenue increases by \$30 per unit for each additional finished desk produced (beyond 300) when the level of production of unfinished desks remains fixed at 250 units.

$$\frac{\partial R}{\partial y} = -0.5 \times 250 - 0.2 \times 300 + 160 = -25$$

Thus, the weekly revenue decreases by \$25 per unit when each additional unfinished desk (beyond 250) is produced and the level of production of finished desks remains fixed at 300 units.

Exercise for Reader

1. Find the first order partial derivatives of the function: $f(x, y, z) = xyz - xe^{yz} + x \ln y$.
2. The total cost of producing 1 unit of a product is $C(x, y) = 30 + 2x + 4y + \frac{xy}{50}$ dollars where x is the cost per pound of raw materials and y is the cost per hour of labor.
 - a) If labor costs are held constant, at what rate will the total cost increase for each increase of \$1 per pound in material cost?
 - b) If material costs are held constant, at what rate will the total cost increase for each \$1 per hour increase in labor costs?
3. The monthly profit (in dollars) of Bond and Barker Department Store depends on the level of inventory x (in thousands of dollars) and the floor space y (in thousands of square feet) available for display of the merchandise, as given by the equation

$$P(x, y) = -0.02x^2 - 15y^2 + xy + 39x + 25y - 20,000$$

Compute $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ when $x = 4000$ and $y = 150$. Interpret your results. Repeat with $x = 5000$ and $y = 150$.