Lecture 23

Learning Objectives

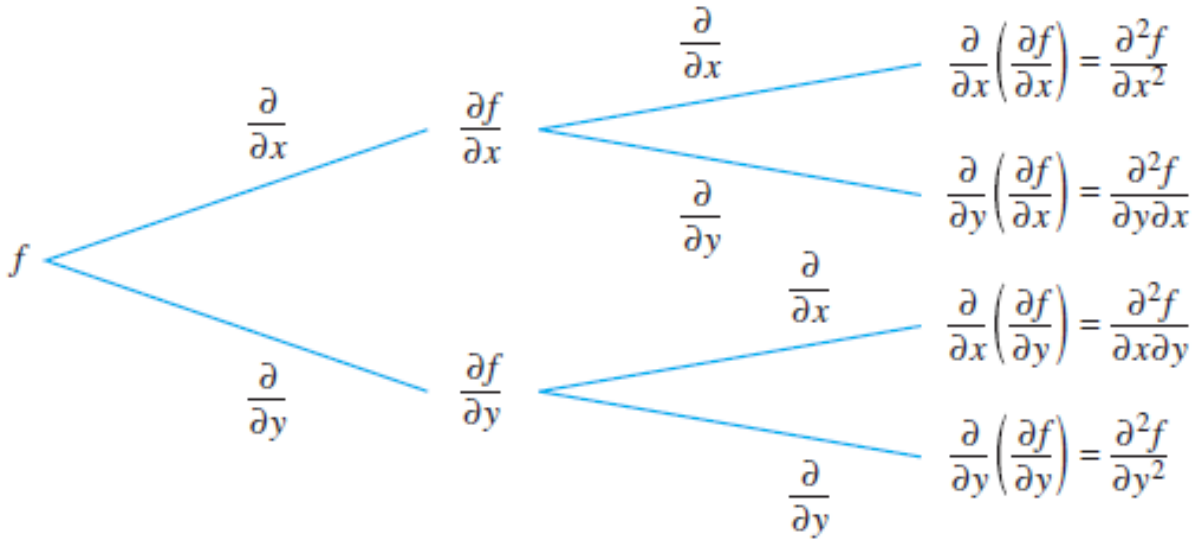
At the end of this class, students should be able to:

- identify second order partial derivatives
- solve related problems

Second-Order Partial Derivatives

The first partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of a function $f(x, y)$ of the two variables x and y are also functions of x and y. we may differentiate each of the functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to obtain the second-order partial derivatives of f.

The following diagram shows the four second-order partial derivatives of the function $f(x, y)$.



Partial Derivative Notations

- $f_x = \frac{\partial f}{\partial x}$
- $f_y = \frac{\partial f}{\partial y}$
- $f_{xx} = \frac{\partial^2 f}{\partial x^2}$
- $f_{yy} = \frac{\partial^2 f}{\partial y^2}$

- $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
- $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

Let us try to understand the concept with the help of the following illustrations.

Illustration 1

Find the second-order partial derivatives of the function: $f(x,\ y) = x^2 y + xy^3$.

Solution

We have $f(x,\ y) = x^2 y + xy^3$ then the first order partial derivatives are:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 y + xy^3) = 2xy + y^3 \text{ and}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 y + xy^3) = x^2 + 3xy^2$$

Now, second first order partial derivatives are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy + y^3)$$

or $$\frac{\partial^2 f}{\partial x^2} = 2y;$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy + y^3)$$

or $$\frac{\partial^2 f}{\partial y \partial x} = 2x + 3y^2;$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^2 + 3xy^2)$$

or $$\frac{\partial^2 f}{\partial x \partial y} = 2x + 3y^2; \text{ and}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x^2 + 3xy^2)$$

or $$\frac{\partial^2 f}{\partial y^2} = 6xy$$

Illustration 2

Find the second-order partial derivatives f_{xx}, f_{xy}, f_{yz} and f_{zz} of the function: $f(x,\ y,\ z) = x^2y^2 + z^2$.

Solution

We have $f(x,\ y,\ z) = x^2y^2 + z^2$ then

$$f_x = \frac{\partial}{\partial x}(x^2y^2 + z^2) = 2xy^2,$$

$$f_{xx} = \frac{\partial}{\partial x}(2xy^2) = 2y^2,$$

$$f_{xy} = \frac{\partial}{\partial y}(2xy^2) = 4xy,$$

$$f_y = \frac{\partial}{\partial y}(x^2y^2 + z^2) = 2x^2y,$$

$$f_{yz} = \frac{\partial}{\partial z}(2x^2y) = 0,$$

$$f_z = \frac{\partial}{\partial z}(x^2y^2 + z^2) = 2z,$$

$$f_{zz} = \frac{\partial}{\partial z}(2z) = 2$$

Illustration 3

If $u = \ln(x^2+y^2)$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution

We have, $u = \ln(x^2+y^2)$ then

$\frac{\partial u}{\partial x} = \frac{1}{(x^2+y^2)}\frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{(x^2+y^2)}$ and

$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$

$= \frac{(x^2+y^2)\times(2) - 2x\times 2x}{(x^2+y^2)^2}$

$= \frac{2y^2-2x^2}{(x^2+y^2)^2}$

Similarly,

$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$

Now,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$

$$= \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2}$$

$$= 0$$

Exercise for Reader

1. Find the second order partial derivatives of each function.

   a) $f(x, y) = 2x^2 + 4y^3 + 5xy$

   b) $f(x, y) = \dfrac{x}{1+y}$

   c) $f(x, y) = x\sqrt{y} + y\sqrt{x}$

   d) $f(x, y) = (x^2 + y^2)^{3/2}$

   e) $f(x, y) = e^{x/y}$

2. Find the partial derivatives: f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, and f_{zy} for $f(x, y, z) = xyz - xe^{yz} + x\ln y$.

3. If $u = \ln(x^2+y^2 + z^2)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2$.