Lecture 24

Learning Objectives

At the end of this class, students should be able to:

- identify maximum and minimum of the given function

Maxima and Minima

A function $u = f(x, y)$ of two variables:

- has a relative maximum at (a, b) if $f(x, y) \le f(a,\ b)$ for all points (x, y) in a region containing (a, b).
- has a relative minimum at (a, b) if $f(x, y) \ge f(a,\ b)$ for all points (x, y) in a region containing (a, b).

To determine the nature of a critical point of a function $f(x, y)$ of two variables, we use the second-order partial derivatives of *f*. The following procedure is adopted for finding and classifying the relative extrema of *f*.

1. Find the critical points of $f(x, y)$ by solving the system of equations $f_x = 0,\ f_y = 0$. Let (a, b) represents a solution of the system of equations.
2. Find $D = f_{xx} . f_{yy} - (f_{xy})^2$.
3. Evaluate the value of D at (a, b). Then
   i) $f(x, y)$ has a relative maximum at (a, b) if $D > 0$ and $f_{xx}(a,b) < 0$.
   ii) $f(x, y)$ has a relative minimum at (a, b) if $D > 0$ and $f_{xx}(a,b) > 0$.
   iii) $f(x, y)$ has neither a maximum nor a minimum at $(a,\ b)$ if $D < 0$. The function has a saddle point at (a, b).
   iv) This test fails if D = 0 at $(a,\ b)$. So, some other technique must be used to solve the problem.

Illustration 1

Find the relative maximum and minimum values of $f(x,\ y) = x^2 + 4y^3 - 12y^2 - 36y + 2$.

Solution

We have $f(x,\ y) = x^2 + 4y^3 - 12y^2 - 36y + 2$ then

$$f_x = 2x,$$

$$f_y = 12y^2 - 24y - 36$$

For critical points, setting $f_x = 0$ and $f_y = 0$, we get

$$2x = 0 \quad \ldots \text{(i)}$$

$$12y^2 - 24y - 36 = 0 \quad \ldots \text{(ii)}$$

Equation (i) gives $x = 0$. Equation (ii) gives

$$y^2 - 2y - 3 = 0$$

or $(y+1)(y-3) = 0$

$\therefore \quad y = -1 \text{ or } 3$

Thus, the required critical points are (0, -1) and (0, 3).

Now, $f_{xx} = 2,\ f_{xy} = 0,$ and $f_{yy} = 24y - 24$.

Here, $D = f_{xx} . f_{yy} - (f_{xy})^2$

$$= 2 \times (24y - 24) - 0$$

$$= 48(y-1)$$

At $(x,\ y) = (0,\ -1)$,

$$D = 48(-1-1) = -96 < 0$$

Thus, the given function has saddle point at $(x,\ y) = (0,\ -1)$.

At $(x,\ y) = (0, 3)$,

$D = 48(3-1) = 96 > 0$ and

$$f_{xx} = 2 > 0$$

Thus, the given function has relative minimum at $(x,\ y) = (0, 3)$.

The minimum value of the function is found by substituting $(x,\ y) = (0, 3)$ in the given function.

$$f(0,\ 3) = 4(3)^3 + (0)^2 - 12(3)^2 - 36 \times 3 + 2$$

$$= -106$$

Illustration 2

Suppose that the surfboard company has developed the yearly profit equation $P(x,\ y) = -22x^2 + 22xy - 11y^2 + 110x - 44y - 23$ where x is the number (in thousands) of standard surfboards produced per year, y is the number (in thousands) of competition surfboards produced per year, and P is profit (in thousands of dollars). How many of each type of board should be produced per year to realize a maximum profit? What is the maximum profit?

Solution

Here, $P(x,\ y) = -22x^2 + 22xy - 11y^2 + 110x - 44y - 23$ then

$$P_x = -44x + 22y + 110 \text{ and}$$

$$P_y = 22x - 22y - 44$$

For critical points, setting $P_x = 0$ and $P_y = 0$, we get

$$-44x + 22y + 110 = 0 \text{ and}$$

$$22x - 22y - 44 = 0$$

Solving above equations, we get $x = 3$ and $y = 1$.

Now, $P_{xx} = -44,\ P_{xy} = 22,$ and $P_{yy} = -22$.

Here, $D = P_{xx}.P_{yy} - (P_{xy})^2$

$$= (-44)\times(-22) - (22)^2 = 484$$

At $(x,\ y) = (3,\ 1)$,

$$D = 484 > 0$$

$$P_{xx} = -44 < 0$$

Thus, the profit will be maximum at $(x,\ y) = (3,\ 1)$.

The maximum profit can be found by substituting $(x,\ y) = (5,\ 1)$ in the given profit function.

$$P(3,\ 1) = -22(3)^2 + 22\times 3\times 1 - 11(1)^2 + 110\times 3 - 44\times 1 + 23 = 120.$$

Thus, a maximum profit of $120,000 is obtained by producing and selling 3,000 standard boards and 1,000 competition boards per year

Exercise for Reader

1. Find the relative maximum and minimum values.

   a) $f(x, y) = x^2 + xy + y^2 - 5y$

   b) $f(x, y) = 2x^2 + y^2 - 4x + 6y + 3$

   c) $f(x, y) = x^3 + y^3 - 3xy$

   d) $f(x, y) = 2x^3 + y^2 - 9x^2 - 4y + 12x - 2$

2. A firm produces two types of earphones per year: *x* thousands of type A and *y* thousands of type B. If the revenue and cost equations for the year are (in millions of dollars) $R(x, y) = 2x + 3y$ and $C(x, y) = x^2 - 2xy + 2y^2 + 6x - 9y + 5$ determine how many of each type of earphone should be produced per year to maximize profit. What is the maximum profit?

3. The total weekly revenue (in dollars) that Acrosonic realizes in producing and selling its bookshelf loudspeaker systems is given by

   $$R(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y$$

   where *x* denotes the number of fully assembled units and *y* denotes the number of kits produced and sold each week. The total weekly cost attributable to the production of these loudspeakers is

   $$C(x, y) = 180x + 140y + 5000$$

   dollars, where *x* and *y* have the same meaning as before. Determine how many assembled units and how many kits Acrosonic should produce per week to maximize its profit.