Lecture 26

Learning Objectives

At the end of this class, students should be able to:

- use integration rules
- identify area under the graph of a function
- understand the concept of definite integral
- solve related problems

26.1 Sum or Difference Rule

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Illustration 1

Evaluate $\int (x^{1/3} - 3x^{-2/3} + 6)dx$

Solution

$$\int (x^{1/3} - 3x^{-2/3} + 6)dx = \int x^{1/3}dx - 3\int x^{-2/3}dx + 6\int dx$$

$$= \frac{3}{4}x^{4/3} - 9x^{1/3} + 6x + c$$

Illustration 2

Evaluate $\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}} \right) dx$

Solution

$$\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}} \right) dx = \frac{1}{2}\int \frac{1}{x}dx - 2\int x^{-2}dx + 3\int x^{-1/2}dx$$

$$= \frac{1}{2}\ln x - \frac{2}{(-1)}x^{-1} + 6x^{1/2} + c$$

$$= \frac{1}{2}\ln x + \frac{2}{x} + 6x^{1/2} + c$$

Illustration 3

Evaluate $\int \left(3e^u + \frac{6}{u} + \ln 2 \right) du$

Solution

$$\int\left(3e^u+\frac{6}{u}+\ln 2\right)du \quad = 3\int e^u du + 6\int\frac{1}{u}du + \ln 2\int du$$

$$= 3e^u + 6\ln u + u\ln 2 + c$$

26.2 Area under the Graph of a Function

If *f* is a continuous function on [*a*, *b*] as shown in the following figure and $f(x)\ge 0$ on [*a*, *b*], then the area between $y = f(x)$ and the *x*-axis from $x = a$ to $x = b$ is given by

$$\text{Shaded Area} = \int_a^b f(x)\,dx$$

Note that if $f(x)\le 0$ on [*a*, *b*], then

$$\int_a^b f(x)dx = \text{- Area (between } f(x) \text{ and } x\text{-axis)}$$

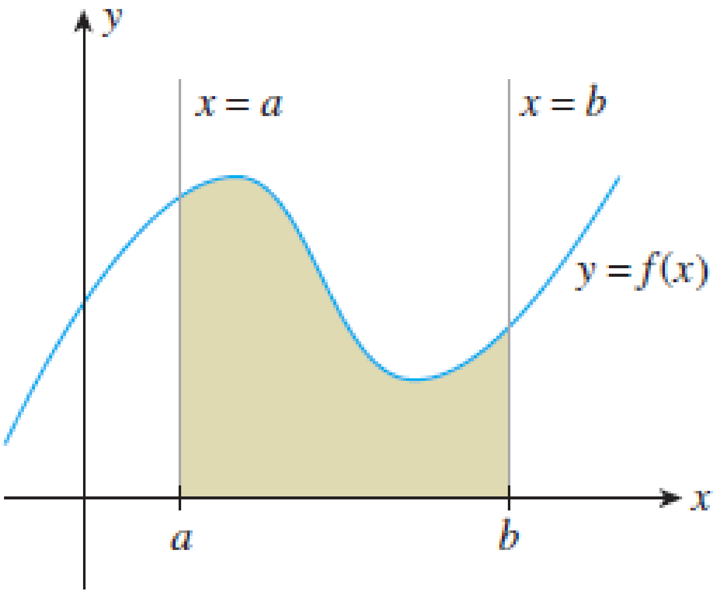

Illustration 4

Let *R* be the region under the graph of $f(x) = 16 - x^2$ on the interval [1, 3]. Find the area of the region R.

Solution

$$\text{Required Area} = \int_1^3 (16 - x^2)\,dx$$

We have to understand the concept of definite integral to find the value of this integral.

26.3 The Definite Integral

Let $f(x)$ be a function that is continuous on the interval [a, b]. Then the definite integral of f on the interval [a, b], denoted by

$$\int_a^b f(x)dx$$

The numbers a and b are called the lower and upper limits of integration, respectively. The process of finding a definite integral is called definite integration.

Fundamental Theorem of Calculus

Let f be a continuous function on the closed interval [a, b]; then the definite integral of f exists on this interval, and

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where F is any function such that $F'(x) = f(x)$ for all x in [a, b].

Thus, we apply the fundamental theorem of calculus by using the following two steps.

Step 1. Integration of $f(x)$: $\int_a^b f(x)dx = F(x)\Big|_a^b$

Step 2. Evaluation of $F(x)$: $F(x)\Big|_a^b = F(b) - F(a)$

Illustration 5

Evaluate $\int_1^3 (16 - x^2)\,dx$

Solution

$$\begin{aligned}
\int_1^3 (16 - x^2)\,dx &= \left(16x - \frac{1}{3}x^3\right)\Big|_1^3 \\
&= \left(16\times 3 - \frac{1}{3}\times 3^3\right) - \left(16\times 1 - \frac{1}{3}\times 1^3\right) \\
&= (48 - 9) - \left(16 - \frac{1}{3}\right) \\
&= 39 - \frac{47}{3} \\
&= \frac{70}{3}
\end{aligned}$$

Exercise for Reader

1. Evaluate the following integrals.

   a) $\int (1+2x+e^{x})dx$

   b) $\int \left(\sqrt{x}+\frac{3}{\sqrt{x}}\right)dx$

2. Evaluate the following integrals.

   a) $\int_{2}^{4} 5\,dx$

   b) $\int_{2}^{4} \frac{1}{x}dx$

   c) $\int_{-1}^{2} (x^{2}-3x+2)dx$

   d) $\int_{2}^{5} e^{2x}dx$