Lecture 27

Learning Objectives

At the end of this class, students should be able to:

- evaluate integrals by using substitution technique

Integration by Substitution

Sometimes, the integrand is not in the appropriate form so that we cannot use the integration formulas directly. In this situation, we need to change the variable by suitable substitution.

We follow the following procedure:

Step 1: Select a substitution that appears to simplify the integrand. In particular, try to select *u* so that *du* is a factor in the integrand.

Step 2: Express the integrand entirely in terms of *u* and *du*, completely eliminating the original variable.

Step 3: Evaluate the new integral if possible.

Step 4: Express the answer found in step 3 in terms of the original variable.

Let us go through the following examples.

Illustration 1

Evaluate $\int (3x+5)^5 \, dx$.

Solution

Let $3x+5=u$ then

$d(3x+5)=du$

or $3dx = du$

$$\left[\because \text{ If } y = f(x) \text{ then } dy = \left\{\frac{d}{dx} f(x)\right\} dx \right]$$

or $dx = du/3$

Then the integral becomes

$$\int (3x+5)^5 \, dx = \int u^5 \, (du/3)$$

$$= \frac{1}{3}\int u^5 du$$

$$= \frac{1}{2} \times \frac{1}{6} u^6 + c$$

$$= \frac{1}{12}(3x+5)^6 + c \qquad [\because\ u = 3x+5]$$

In general,

1. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + k$; for $n \neq -1$, and

2. $\int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln|(ax+b)| + k$

We can use these results as formulas.

Illustration 2

Evaluate $\int (5x+2)^{-3}\,dx$.

Solution

$$\int (5x+2)^{-3}\,dx = \frac{(5x+2)^{-2}}{5\times(-2)} + c$$

$$\left[\because\ \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + k\right]$$

$$= -\frac{1}{12}(5x+2)^{-2} + c$$

Illustration 3

Evaluate $\int \frac{1}{(2x+3)}\,dx$.

Solution

$$\int \frac{1}{(2x+3)}\,dx = \frac{1}{2}\ln(2x+3) + c$$

$$\left[\because\ \int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln|(ax+b)| + k\right]$$

Illustration 4

Evaluate $\int 2x(x^2+5)^6\,dx$.

Solution

Let $x^2 + 5 = u$ then

or $2xdx = du$

or $dx = \dfrac{du}{2x}$

Then the integral becomes

$$\int 2x(x^2+5)^6 dx \quad = \int 2x \times u^6 \times \frac{du}{2x}$$

$$= \int u^6 du$$

$$= \frac{1}{7}u^7 + c$$

$$= \frac{1}{7}(x^2+5)^7 + c$$

Illustration 5

Evaluate $\displaystyle\int \frac{(\ln x)^2}{3x} dx$.

Solution

Let $\ln x = u$ then

or $\dfrac{1}{x}dx = du$

or $dx = x\,du$

Then the integral becomes

$$\int \frac{(\ln x)^2}{3x} dx \quad = \int \frac{(u)^2}{3x} \times x\,du$$

$$= \frac{1}{3}\int u^2 du$$

$$= \frac{1}{3} \times \frac{1}{3}u^3 + c$$

$$= \frac{1}{9}(\ln x)^3 + c$$

Illustration 6

Evaluate $\displaystyle\int x^2 e^{x^3-1} dx$

Solution

Let $x^3 - 1 = u$ then

$3x^2 dx = du$

or $\quad dx = \dfrac{du}{3x^2}$

Then the integral becomes

$$\int x^2 e^{x^3-1} dx = \int x^2 e^u \times \frac{du}{3x^2}$$

$$= \frac{1}{3}\int e^u du$$

$$= \frac{1}{3}e^u + c$$

$$= \frac{1}{3}e^{x^3-1} + c$$

Exercise for Reader

Evaluate the following integrals.

a) $\displaystyle\int \frac{4}{(2x+3)^3}dx$ b) $\displaystyle\int \frac{4x}{(2x^2+3)^3}dx$

c) $\displaystyle\int e^{2x}(e^{2x}+1)^5 dx$ d) $\displaystyle\int \frac{1}{x(\ln x)^2}dx$

e) $\displaystyle\int (3x^2-1)e^{x^3-x}dx$ f) $\displaystyle\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$