

Lecture 28

Learning Objectives

At the end of this class, students should be able to:

- evaluate integrals by using substitution technique
- evaluate integrals by using partial fractions

Illustration 1

Evaluate $\int_0^2 \frac{6x}{\sqrt{x^2 + 5}} dx$

Solution

Let $x^2 + 5 = u$ then

$$2x dx = du$$

or $dx = \frac{du}{2x}$

When $x = 0$, $u = 0^2 + 5 = 5$ and

when $x = 2$, $u = 2^2 + 5 = 9$.

Then the given integral becomes

$$\begin{aligned} \int_0^2 \frac{6x}{\sqrt{x^2 + 5}} dx &= \int_5^9 \frac{6x}{\sqrt{u}} \times \frac{du}{2x} \\ &= 3 \int_5^9 u^{-1/2} du \\ &= 3 \times 2u^{1/2} \Big|_5^9 \\ &= 6(9^{1/2} - 5^{1/2}) \\ &= 6(3 - \sqrt{5}) \end{aligned}$$

Illustration 2

Evaluate $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution

Let $\sqrt{x} = u$ then

$$\frac{1}{2\sqrt{x}} dx = du$$

or $dx = 2\sqrt{x} du$

When $x = 0$, $u = \sqrt{0} = 0$ and when $x = 4$, $u = \sqrt{4} = 2$.

Then the integral becomes

$$\begin{aligned} \int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_0^2 \frac{e^u}{\sqrt{x}} \times 2\sqrt{x} du \\ &= 2 \int_0^2 e^u du \\ &= 2e^u \Big|_0^2 \\ &= 2(e^2 - e^0) \\ &= 2(e^2 - 1) \end{aligned}$$

Integration by Partial Fractions

There are rational functions which cannot be integrated by the techniques presented earlier. When this occurs, one possibility is to break it into partial fractions, and then find the integrals of these fractions.

Case I

Whenever numerator is of equal or higher degree than the denominator.

We divide the numerator by denominator until a remainder is obtained which is of lower degree than the denominator, i.e.,

$$\frac{p(x)}{q(x)} = \text{Quotient} + \frac{\text{Remainder}}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and the degree of p is greater than or equal to that of q .

Illustration 3

Evaluate $\int \frac{x^2+3x+5}{x+1} dx$.

Solution

Here, $\frac{x^2+3x+5}{x+1} = (x+2) + \frac{3}{x+1}$, then

$$\int \frac{x^2+3x+5}{x+1} dx = \int \left[(x+2) + \frac{3}{x+1} \right] dx$$

$$\begin{aligned}
&= \int (x + 2)dx + 3 \int \frac{1}{x+1} dx \\
&= \left(\frac{x^2}{2} + 2x\right) + 3 \ln(x + 1) + c
\end{aligned}$$

Case II

When the numerator is of lower degree than the denominator and the linear factors in the denominator are non-repeated, we write the given fraction in the following form and find the values of A and B.

$$\frac{p(x)}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Illustration 4

Evaluate $\int \frac{5x}{x^2+3x+2} dx$.

Solution

Here, the denominator = $x^2 + 3x + 2 = (x + 1)(x + 2)$

Thus, $\frac{5x}{x^2+3x+2} = \frac{5x}{(x+1)(x+2)}$

Let $\frac{5x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

or, $\frac{5x}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$

Equating both sides, we get

$$A(x + 2) + B(x + 1) = 5x$$

$$A(x + 2) + B(x + 1) = 5x$$

Putting $x = -2$, we get

$$B(-1) = 5(-2),$$

or $B = 10$

Again, putting $x = -1$, we get $A = -5$

Thus, $\frac{5x}{(x+1)(x+2)} = \frac{-5}{(x+1)} + \frac{10}{(x+2)}$

Now,

$$\begin{aligned}
\int \frac{5x}{x^2+3x+2} dx &= \int \left[\frac{-5}{(x+1)} + \frac{10}{(x+2)} \right] dx \\
&= -5 \ln(x + 1) + 10 \ln(x + 2) + c
\end{aligned}$$

Exercise for Reader

Evaluate the following integrals.

a) $\int_0^1 3x^2 e^{x^3} dx$

b) $\int_0^2 \frac{4x}{\sqrt{1+2x^2}} dx$

c) $\int_1^e \frac{(\ln x)^2}{x} dx$

d) $\int_{-1}^0 \frac{e^{-x}}{(1+e^{-x})^2} dx$

e) $\int_1^4 \frac{(\sqrt{x}-1)^{3/2}}{\sqrt{x}} dx$

f) $\int \frac{x^2+3x+3}{x+1} dx$

g) $\int \frac{5x-7}{x^2-2x-15} dx$