Lecture 30

Learning Objectives

At the end of this class, students should be able to:

- solve the problems with boundary conditions
- determine the corresponding function from its marginal function

30.1 Initial Conditions or Boundary Conditions

We get constant of integration c when we find integral. This constant of integration c may be of interest in some applications. In many problems, an initial condition ($y = y_0$ when $x = 0$) or a boundary condition ($y = y_0$ when $x = x^*$) is given which uniquely determines the constant of integration.

Illustration 1

The management of Lorimar Watch Company has determined that the daily marginal revenue function associated with producing and selling their travel clocks is given by $R'(x) = -0.009x + 12$where x denotes the number of units produced and sold and $R'(x)$ is measured in dollars/unit. Determine the revenue function $R(x)$ associated with producing and selling these clocks.

Solution

We have marginal revenue $R'(x) = -0.009x + 12$ then the total revenue function is

$$R(x) = \int(-0.009x + 12)dx$$

or, $R(x) = -0.0045x^2 + 12x + c$

When $x = 0$, $R = 0$, so

$$0 = -0.0045\times0^2 + 12\times0 + c$$

$\therefore$ $c = 0$

Thus, the required total revenue function is

$$R(x) = -0.0045x^2 + 12x$$

Illustration 2

A company determines that the marginal cost, $C'(x)$, of producing the x^{th} unit of a product is given by $C'(x) = x^3 - 2x$.

Find the total-cost function, $C(x)$, assuming that $C(x)$ is in dollars and the fixed costs are \$5,000.

Solution

Here, marginal cost is

$$C'(x) = x^3 - 2x$$

Then that the total cost function is

$$C(x) = \int (x^3 - 2x)dx$$

or, $C(x) = \frac{1}{4}x^4 - x^2 + k$

According to question, when $x = 0$, $C(x) = 5{,}000$, then

$$5{,}000 = \frac{1}{4} \times 0^4 - 0^2 + k$$

$\therefore$ $k = 5{,}000$

Thus, the required cost function is

$$C(x) = \frac{1}{4}x^4 - x^2 + 5{,}000$$

30.2 Consumers' Surplus and Producers' surplus

Suppose $p = D(x)$ is the demand function. Suppose the market price of a particular commodity has been fixed at $\overline{p}$ and consumers will buy $\overline{x}$ units at that price. Then the difference between the consumers' willingness to pay for $\overline{x}$ units and the amount they actually pay represents a perceived advantage to the consumer that economists call consumers' surplus.

Mathematically,

$$\text{CS} = \int_0^{\overline{x}} D(x)dx - \overline{p} \times \overline{x}$$

Suppose $p = S(x)$ is the producers' supply function for the commodity. If $\overline{x}$ units of a commodity are sold at a price of $\overline{p}$ then the difference between the consumers' actual expenditure for $\overline{x}$ units and the total amount the producer receives when for $\overline{x}$ units are sold is called the producers' surplus.

Mathematically,

$$PS = \overline{p} \times \overline{x} - \int_0^{\overline{x}} S(x)dx$$

Illustration 3

Find the equilibrium price and then find the consumers' surplus and producers' surplus at the equilibrium price level, if $p = D(x) = 20 - 0.05x$ and $p = S(x) = 2 + 0.0002x^2$.

Solution

For equilibrium point, $D(x) = S(x)$

i.e. $20 - 0.05x = 2 + 0.0002x^2$

or $0.0002x^2 + 0.05x - 18 = 0$

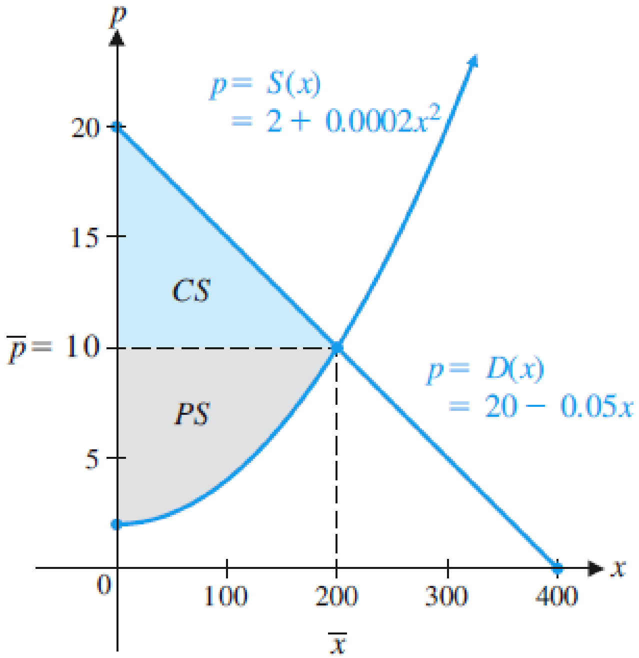
or $x^2 + 250x - 90{,}000 = 0$

$\therefore$ $x = -450,\ 200$

Since x cannot be negative, the only solution is x = 200. The equilibrium price can be determined by using $D(x)$ or $S(x)$.

Now $\bar{p} = D(200) = 20 - 0.05 \times 200 = 10$

Thus, the equilibrium price is $\bar{p} = 10$ and the equilibrium quantity is $\bar{x} = 200$. This can be seen in the following figure.



The consumers' surplus is given by

$$CS = \int_0^{\bar{x}} D(x)dx - \bar{p} \times \bar{x}$$

$$= \int_0^{200} (20 - 0.05x)dx - 10 \times 200$$

$$= (20x - 0.025x^2)\Big|_0^{200} - 2{,}000$$

$$= \$1{,}000$$

Similarly, the producers' surplus is given by

$$PS = \bar{p} \times \bar{x} - \int_0^{\bar{x}} S(x)dx$$

$$= 10 \times 200 - \int_0^{200} (2 + 0.0002x^2)dx$$

$$= 2{,}000 - \left(2x + \frac{0.0002}{3}x^3\right)\Big|_0^{200}$$

$$= \$1{,}067$$

Exercise for Reader

1. A company determines that the marginal cost, $C'(x)$,of producing the x^{th} unit of a product is given by $C'(x) = x^3 - x$ Find the total-cost function, C, assuming that C(x) is in dollars and that fixed costs are \$6500.
2. A company determines that the marginal revenue, $R'(x)$ in dollars, from selling the x^{th} unit of a product is given by $R'(x) = x^2 - 1$. Find the total-revenue function, $R(x)$ assuming that $R(0) = 0$.
3. If the marginal cost for a product is $MC = 4x + 50$, and the total cost of producing 20 units is \$2000, find the total cost function.
4. A certain firm's marginal cost for a product is $MC = 6x + 60$, its marginal revenue is $MR = 180 - 2x$, and its total cost of production of 10 items is \$1000.
   a) Find the optimal level of production.
   b) Find the profit function.
   c) Find the profit or loss at the optimal level of production.
5. The demand function for a certain brand of CD is given by $p = -0.01x^2 - 0.2x + 8$ where p is the wholesale unit price in dollars and x is the quantity demanded each week, measured in units of a thousand. Determine the consumers' surplus if the wholesale market price is set at \$5/disc.
6. The supplier of the portable hair dryers will make x hundred units of hair dryers available in the market when the wholesale unit price is $p = \sqrt{36 + 1.8x}$ dollars. Determine the producers' surplus if the wholesale market price is set at \$9/unit.
7. Find the total amount of money consumers are willing to spend to get $\bar{x}$ units of the commodity.
   a) $D(x) = 2(64 - x^2)$ dollars per unit; $\bar{x} = 6$ units.
   b) $D(x) = \dfrac{400}{0.5x + 2}$ dollars per unit; $\bar{x} = 12$ units.
8. Compute the corresponding consumers' surplus.
   a) $D(x) = 2(64 - x^2)$; $\bar{x} = 3$ units.
   b) $D(x) = 40e^{-0.05x}$; $\bar{x} = 5$ units.
9. The management of the Titan Tire Company has determined that the quantity demanded x of their Super Titan tires/week is related to the unit price p by the relation $p = 144 - x^2$ where p is measured in dollars and x is measured in units of a thousand. Titan will make x units of the tires available in the market if the unit price is $p = 48 + (1/2)x^2$ dollars. Determine the consumers' surplus and the producers' surplus when the market unit price is set at the equilibrium price.
10. The quantity demanded x (in units of a hundred) of the Mikado miniature cameras/week is related to the unit price p (in dollars) by $p = 80 - 0.2x^2$ and the quantity x (in units of a hundred) that the supplier is willing to make available in the market is related to the unit

price *p* (in dollars) by $p = 40 + x + 0.1x^2$ .If the market price is set at the equilibrium price, find the consumers' surplus and the producers' surplus.