Lecture 33

Learning Objectives

At the end of this class, students should be able to:

- evaluate the value of determinant

Determinant

The determinant of a matrix *A* of order *n* × *n* with numbers as elements is a number, assigned to matrix *A* by the following rule:

$$\det A = |A| = \sum_{j=1}^{n} (-1)^{1+j}.a_{1j}.M_{1j} \qquad \ldots \text{(i)}$$

Where, M_{1j} is a known as minor of an element a_{1j} and $(-1)^{1+j}.M_{1j}$ is known as cofactor of an element a_{1j}.

For example, suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3.

We get the following matrix of order 2×2 if we delete the first row and the first column of this matrix,

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

The determinant of this matrix is

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Which is called the *minor* of the element a_{11}, denoted by M_{11}.

Thus, $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Similarly, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,

$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

In general, the minor of element a_{ij} is denoted by M_{ij}.

The *cofactor* of the element a_{ij} is denoted by A_{ij}, and defined by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Thus,

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ and so on.}$$

When n = 1 then matrix $A = [a_{11}]$, thus from the formula (i), we get

$$|A| = a_{11}$$

When n = 2 then matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, thus from the formula (i), we get

$$|A| = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12}$$

i.e. $$|A| = a_{11}a_{22} - a_{12}a_{21}$$

When n = 3 then matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, thus from the formula (i), we get

$$|A| = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + (-1)^{1+3} a_{13} M_{13}$$

i.e. $$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

i.e. $$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Illustration 1

Evaluate the following determinants:

a) |4|

b) |–5|

c) $\begin{vmatrix} 6 & 5 \\ 3 & -7 \end{vmatrix}$

d) $\begin{vmatrix} 2 & 3 & 6 \\ 1 & 0 & 7 \\ 4 & 5 & 8 \end{vmatrix}$

Solution

a) $|4| = 4$

b) $|-5| = -5$

c) $\begin{vmatrix} 6 & 5 \\ 3 & -7 \end{vmatrix} = 6\times(-7)-3\times5 \;=-57$

d) $\begin{vmatrix} 2 & 3 & 6 \\ 1 & 0 & 7 \\ 4 & 5 & 8 \end{vmatrix} = 2(0\times8-5\times7)-3(1\times8-4\times7)+6(1\times5-4\times0)$

$= 2(0-35)-3(8-28)+6(5-0)$

$= -70+60+30$

$= 20$

Exercise for Reader

Evaluate the following determinants:

a) $|-7|$

b) $\begin{vmatrix} 2 & -5 \\ 3 & -4 \end{vmatrix}$

c) $\begin{vmatrix} 5 & 2 & 3 \\ 0 & 3 & -1 \\ 6 & 4 & 7 \end{vmatrix}$