Lecture 34

Learning Objectives

At the end of this class, students should be able to:

- apply properties of determinant

Properties of Determinant

The following properties of the determinant are useful to evaluate the determinant.

1. The value of the determinant remains unaltered by changing its rows and columns.

   i.e.,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If any two adjacent rows (or columns) are interchanged, the sign of the determinant will be changed.

   For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$$

3. If any two rows (or two columns) of a determinant are identical, then the value of the determinant is zero.

   For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

4. If all the elements of any one row (or column) are zero, then the value of the determinant is zero.

   For example,

$$\begin{vmatrix} a_1 & 0 & c_1 \\ a_1 & 0 & c_1 \\ a_3 & 0 & c_3 \end{vmatrix} = 0$$

5. If all the elements of any row (or column) are multiplied by a constant k, then the value of the determinant is multiplied by k.

   For example,

$$\begin{vmatrix} a_1 & b_1 & kc_1 \\ a_2 & b_2 & kc_2 \\ a_3 & b_3 & kc_3 \end{vmatrix} = \text{k} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

6. If a row (or column) or a multiple of any row (or column) is added to or subtracted from any other row (or column), the value of the determinant is unaltered.

   For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 + kc_1 & c_1 \\ a_2 & b_2 + kc_2 & c_2 \\ a_3 & b_3 + kc_3 & c_3 \end{vmatrix}$$

7. If A is the upper (or lower) triangular matrix, then $|\text{A}|$ is equal to the product of the main diagonal entries.

   For example

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

8. The determinant of the product of two matrices of order n is the product of their determinants.

   That is, $|AB| = |A||B|$

Illustration 1

Without expanding to any stage, prove that:

$$\begin{vmatrix} 115 & 106 & 97 \\ 10 & 1 & -8 \\ 106 & 97 & 88 \end{vmatrix} = 0$$

Solution

Here, the left-hand side is $\begin{vmatrix} 115 & 106 & 97 \\ 10 & 1 & -8 \\ 106 & 97 & 88 \end{vmatrix}$

Applying $C_2 \rightarrow C_2 - \frac{1}{2}(C_1 + C_3)$, we get

$$\begin{vmatrix} 115 & 0 & 97 \\ 10 & 0 & -8 \\ 106 & 0 & 88 \end{vmatrix}$$

= 0

[∵ all the elements of second column are zero]

Illustration 2

Without expanding to any stage, prove that:

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$$

Solution

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & bca \\ c & c^2 & cab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= 0$$

Illustration 3

Prove that:

$$\begin{vmatrix} \boldsymbol{a+b+2c} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{b+c+2a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{c+a+2b} \end{vmatrix} = \boldsymbol{2(a+b+c)^3}$$

Solution

LHS $\begin{vmatrix} \boldsymbol{a+b+2c} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{b+c+2a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{c+a+2b} \end{vmatrix}$

Applying $\boldsymbol{C_1 \rightarrow C_1 + C_2 + C_3}$, *we get*

$$\begin{vmatrix} \boldsymbol{2a+2b+2c} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{2a+2b+2c} & \boldsymbol{b+c+2a} & \boldsymbol{b} \\ \boldsymbol{2a+2b+2c} & \boldsymbol{a} & \boldsymbol{c+a+2b} \end{vmatrix}$$

Taking 2($\boldsymbol{a+b+c}$*) common from* $\boldsymbol{C_1}$, *we get*

$$\boldsymbol{2(a+b+c)} \begin{vmatrix} \boldsymbol{1} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{1} & \boldsymbol{b+c+2a} & \boldsymbol{b} \\ \boldsymbol{1} & \boldsymbol{a} & \boldsymbol{c+a+2b} \end{vmatrix}$$

Applying $\boldsymbol{R_2 \rightarrow R_2 - R_1}$ and $\boldsymbol{R_3 \rightarrow R_3 - R_1}$, we get

$$\boldsymbol{2(a+b+c)} \begin{vmatrix} \boldsymbol{1} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{0} & \boldsymbol{b+c+a} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{c+a+b} \end{vmatrix}$$

Expanding along C_1, we get

$$\boldsymbol{= 2(a+b+c)[1\{(a+b+c)^2 - 0\} - 0 + 0]}$$

$$\boldsymbol{= 2(a+b+c)^3}$$

Illustration 4

Solve: $\begin{vmatrix} x & -5 \\ -3 & x-2 \end{vmatrix} = 0$

Solution

Here $\begin{vmatrix} x & -5 \\ -3 & x-2 \end{vmatrix} = 0$

or $x^2 - 2x - 15 = 0$

or $(x-5)(x+3) = 0$

$\therefore$ $x = -3, 5$

Exercise for Reader

1. Without expanding to any stage, prove that:

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$$

2. Determine values of t for which the following determinant is equal to zero.

$$\begin{vmatrix} 3 & 1 & 2 \\ 2+2t & 0 & 4 \\ 1 & 2-t & 0 \end{vmatrix}$$

3. Show that:

a) $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

b) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$