

Lecture 35

Learning Objectives

At the end of this class, students should be able to:

- find inverse matrix

The Inverse Matrix

Let A be a square matrix of size n . A square matrix A^{-1} of size n such that

$$AA^{-1} = A^{-1}A = I \text{ is called the inverse of } A.$$

A square matrix A is said to be singular if $|A| = 0$. A square matrix A is said to be non-singular if $|A| \neq 0$.

Not every square matrix has an inverse. A square matrix has an inverse if it is nonsingular.

We use the following formula to find an inverse

$$A^{-1} = \frac{Adj A}{|A|}, \text{ provided that } |A| \neq 0.$$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$\text{the matrix } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \text{ is called the matrix of cofactors.}$$

$$\text{Where } A_{ij} = (-1)^{i+j} M_{ij}$$

The transpose of this matrix of cofactors is called the adjoint of the matrix A and is denoted by $Adj A$.

$$\text{Thus, } Adj A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Illustration 1

$$\text{For the matrix } A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 7 & 3 & -1 \end{bmatrix} \text{ find } A^{-1}, \text{ if it exists.}$$

Solution

$$\text{Here, } |A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 7 & 3 & -1 \end{vmatrix} = 9 \neq 0$$

Since $|A| \neq 0$ therefore A^{-1} exists.

Now, let us find cofactors

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 7 & -1 \end{vmatrix} = 9$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 7 & 3 \end{vmatrix} = 6$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 7 & -1 \end{vmatrix} = 6$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 3 \end{vmatrix} = 11$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

Thus, the matrix of cofactors = $\begin{bmatrix} -3 & 9 & 6 \\ -1 & 6 & 11 \\ 2 & -3 & -4 \end{bmatrix}$ then

$$\text{Adjoint } (A) = \begin{bmatrix} -3 & -1 & 2 \\ 9 & 6 & -3 \\ 6 & 11 & -4 \end{bmatrix}$$

$$\text{Now } A^{-1} = \frac{\text{Adj}(A)}{|A|}$$

$$\begin{aligned} &= \frac{1}{9} \begin{bmatrix} -3 & -1 & 2 \\ 9 & 6 & -3 \\ 6 & 11 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -1/3 & -1/9 & 2/9 \\ 1 & 2/3 & -1/3 \\ 2/3 & 11/9 & -4/9 \end{bmatrix} \end{aligned}$$

Exercise for Reader

Find A^{-1} for the following matrices.

a) $A = \begin{bmatrix} 6 & -1 \\ -3 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 7 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 0 \\ 1 & 9 & 6 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 3 & 0 \\ 8 & 2 & 5 \\ 4 & 0 & 6 \end{bmatrix}$