Lecture 36

Learning Objectives

At the end of this class, students should be able to:

- solve the system of linear equations by using Cramer's rule

Cramer's Rule

This is the method of solving systems of linear equations using determinants. In order to apply this method, the system must have the same number of equations as variables, that is, the coefficient matrix of the system must be square.

(3 × 3) System

Let us consider the system of three linear equations in 3 variables

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

We need to identify the determinants: Δ, Δ_1, Δ_2, $\boldsymbol{and}$ Δ_3 to solve this system.

Where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \text{ and}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

There might be three different situations:

1. If $\Delta \neq 0$, the given system of equation has a unique solution.
2. If $\Delta = 0$ $\boldsymbol{and}$ $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then the system has infinitely many solutions.
3. If $\Delta = 0$ and any $\Delta_j \neq 0$, then then the system has no solution.

If $\Delta \neq \mathbf{0}$, the unique solution is given by

$$\boldsymbol{x_1 = \frac{\Delta_1}{\Delta}}\,,\quad \boldsymbol{x_2 = \frac{\Delta_2}{\Delta}}\,,\ \boldsymbol{and\ x_3 = \frac{\Delta_3}{\Delta}}$$

(2 × 2) System

Let us consider the system of two linear equations in 2 variables:

$$\boldsymbol{a_{11}x_1 + a_{12}x_2 = b_1}$$

$$\boldsymbol{a_{21}x_1 + a_{22}x_2 = b_2}$$

The solution of this system can be found by using formula:

$$\boldsymbol{x_1 = \frac{\Delta_1}{\Delta}}\,,\quad \boldsymbol{x_2 = \frac{\Delta_2}{\Delta}}$$

Where,

$$\Delta = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{vmatrix},\ \Delta_1 = \begin{vmatrix} \boldsymbol{b_1} & \boldsymbol{a_{12}} \\ \boldsymbol{b_2} & \boldsymbol{a_{22}} \end{vmatrix},\ \Delta_2 = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{b_1} \\ \boldsymbol{a_{21}} & \boldsymbol{b_2} \end{vmatrix}$$

Illustration

Solve the following system of equations by using Cramer's rule.

$$x_1 - 2x_2 + x_3 = -4$$

$$2x_1 + x_2 - x_3 = 5$$

$$3x_1 + 2x_2 + 4x_3 = 3$$

Solution

Here

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= 1(4+2) - (-2)(8+3) + 1(4-3) = 29$$

$$\Delta_1 = \begin{vmatrix} -4 & -2 & 1 \\ 5 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= -4(4+2) - (-2)(20+3) + 1(10-3) = 29$$

$$\Delta_2 = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & -1 \\ 3 & 3 & 4 \end{vmatrix}$$

$$= 1(20+3) - (-4)(8+3) + 1(6-15) = 58$$

Similarly,

$$\Delta_3 = \begin{vmatrix} 1 & -2 & -4 \\ 2 & 1 & 5 \\ 3 & 2 & 3 \end{vmatrix}$$

$$= -29$$

Now, according to Cramer's rule

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{29}{29} = 1,$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{58}{29} = 2,$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{-29}{29} = -1$$

Thus, the solution of the given system is $x_1 = 1, x_2 = 2, \text{and } x_3 = -1$.

Illustration

A brokerage house offers three stock portfolios. Portfolio I consists of 2 blocks of common stock and 1 municipal bond. Portfolio II consists of 4 blocks of common stock, 2 municipal bonds, and 3 blocks of preferred stock. Portfolio III consists of 7 blocks of common stock, 3 municipal bonds, and 3 blocks of preferred stock. A customer wants 21 blocks of common stock, 10 municipal bonds, and 9 blocks of preferred stock. How many units of each portfolio should be offered?

Solution

Let us tabulate the given information.

Stock	Portfolio I	Portfolio II	Portfolio III	Expected
Common Stock	2	4	7	21
Municipal Bond	1	2	3	10
Preferred Stock	0	3	3	9

Let x_1 be the units of portfolio I, x_2 be the units of portfolio II, and x_3 be the units of portfolio III, then we can develop the following equations.

Common Stock: $2x_1 + 4x_2 + 7x_3 = 21$

Municipal Bond: $x_1 + 2x_2 + 3x_3 = 10$

Preferred Stock: $3x_2 + 3x_3 = 9$

Here,

$\Delta = 3$, $\Delta_1 = 9$, $\Delta_2 = 6$, and $\Delta_3 = 3$

Now, according to Cramer's rule

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{9}{3} = 3,$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{6}{3} = 2,$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{3}{3} = 1$$

Exercise for Reader

1. Solve the following systems:

a) $$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 4x_2 &= 10 \end{aligned}$$

b) $$\begin{aligned} 5x_1 - 2x_2 &= 6 \\ 3x_1 + 3x_2 &= 12 \end{aligned}$$

c) $$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

d) $$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ 2x_1 + x_2 + x_3 &= 7 \\ 2x_1 + 2x_2 + x_3 &= 10 \end{aligned}$$

2. A manufacturer of table saws has three models, Deluxe, Premium, and Ultimate, which must be painted, assembled, and packaged for shipping. The table gives the number of hours required for each of these operations for each type of table saw.

   a) If the manufacturer has 96 hours available per day for painting, 156 hours for assembly, and 37 hours for packaging, how many of each type of saw can be produced each day?

   b) If 8 more hours of painting time become available, find the new production strategy, and tell how it is related to the inverse matrix used in part (a).

3. A company offers three mutual fund plans for its employees. Plan I consists of 4 blocks of common stock and 2 municipal bonds. Plan II consists of 8 blocks of common stock, 4 municipal bonds, and 6 blocks of preferred stock. Plan III consists of 14 blocks of common stock, 6 municipal bonds, and 6 blocks of preferred stock. An employee wants to combine these plans so that she has 84 blocks of common stock, 40 municipal bonds, and 36 blocks of preferred stock. How many units of each plan does she need?