Lecture 34

Learning Objectives

At the end of this class, students should be able to:

- solve the system of linear equations

Systems of Linear Equations

The important use of matrices and determinants is in the solution of a system of linear equations.

Let us consider the system of linear equations in *n* variables

$$\boldsymbol{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1}$$

$$\boldsymbol{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2}$$

.

.

.

$$\boldsymbol{a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m}$$

Where a_{11}, a_{12}, a_{13} etc. are real numbers which are called coefficients of the system. The *b*'s on the right-hand side are also the real numbers.

The matrix form of the above system of equations is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

which is equivalent to $AX = B$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A system of equations may or may not have a solution. If there exists at least one solution, the system of equations is said to be consistent. Otherwise it is said to be inconsistent.

If *A* be a non-singular square matrix of size *n* and $AX = B$ is a linear system of *n* equations in *n* unknowns then $X = A^{-1}B$ is the unique solution of the system.

Illustration

Solve the following system of equations by using inverse of a matrix

$$\boldsymbol{x_1 + 2x_2 - x_3 = 4}$$

$\boldsymbol{2x_1 + x_3 = 8}$

$\boldsymbol{7x_1 + 3x_2 - x_3 = 19}$

Solution

We may write the given systems of equations in the form $\boldsymbol{AX = B}$.

Where,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 7 & 3 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix}$$

The inverse of *A* is

$$A^{-1} = \begin{bmatrix} -1/3 & -1/9 & 2/9 \\ 1 & 2/3 & -1/3 \\ 2/3 & 11/9 & -4/9 \end{bmatrix}$$

Using this result, we find that the solution of the system is

$$X = A^{-1}B$$

$$= \begin{bmatrix} -1/3 & -1/9 & 2/9 \\ 1 & 2/3 & -1/3 \\ 2/3 & 11/9 & -4/9 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 19 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

or, $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Thus, $x_1 = 2,\ x_2 = 3, and\ x_3 = 4$

Cramer's Rule

This is the method of solving systems of linear equations using determinants. In order to apply this method, the system must have the same number of equations as variables, that is, the coefficient matrix of the system must be square.

(3 × 3) System

Let us consider the system of three linear equations in 3 variables

$\boldsymbol{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1}$

$\boldsymbol{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2}$

$\boldsymbol{a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3}$

We need to identify the determinants: $\Delta,\ \Delta_1,\ \Delta_2,\ \boldsymbol{and}\ \Delta_3$ to solve this system.

Where

$$\Delta = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}} \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \text{ and}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

There might be three different situations:

1. If $\Delta \neq \mathbf{0}$, the given system of equation has a unique solution.
2. If $\Delta = \mathbf{0}$ ***and*** $\Delta_1 = \Delta_2 = \Delta_3 = \mathbf{0}$, then the system has infinitely many solutions.
3. If $\Delta = \mathbf{0}$ and any $\Delta_j \neq \mathbf{0}$, then then the system has no solution.

If $\Delta \neq \mathbf{0}$, the unique solution is given by

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \text{ and } x_3 = \frac{\Delta_3}{\Delta}$$

(2 × 2) System

Let us consider the system of two linear equations in 2 variables:

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

The solution of this system can be found by using formula:

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}$$

Where,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Illustration

Solve the following system of equations by using Cramer’s rule.

$$x_1 - 2x_2 + x_3 = -4$$

$$2x_1 + x_2 - x_3 = 5$$

$$3x_1 + 2x_2 + 4x_3 = 3$$

Solution

Here

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= 1(4+2)-(-2)(8+3)+1(4-3)=29$$

$$\Delta_1 = \begin{vmatrix} -4 & -2 & 1 \\ 5 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= -4(4+2)-(-2)(20+3)+1(10-3)=29$$

$$\Delta_2 = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & -1 \\ 3 & 3 & 4 \end{vmatrix}$$

$$= 1(20+3)-(-4)(8+3)+1(6-15)=58$$

Similarly,

$$\Delta_3 = \begin{vmatrix} 1 & -2 & -4 \\ 2 & 1 & 5 \\ 3 & 2 & 3 \end{vmatrix}$$

$$= -29$$

Now, according to Cramer's rule

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{29}{29} = 1,$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{58}{29} = 2,$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{-29}{29} = -1$$

Thus, the solution of the given system is $x_1 = 1, x_2 = 2, \text{and}\, x_3 = -1$.

Illustration

A brokerage house offers three stock portfolios. Portfolio I consists of 2 blocks of common stock and 1 municipal bond. Portfolio II consists of 4 blocks of common stock, 2 municipal bonds, and 3 blocks of preferred stock. Portfolio III consists of 7 blocks of common stock, 3 municipal bonds, and 3 blocks of preferred stock. A customer wants 21 blocks of common stock, 10 municipal bonds, and 9 blocks of preferred stock. How many units of each portfolio should be offered?

Solution

Let us tabulate the given information.

Stock	Portfolio I	Portfolio II	Portfolio III	Expected
Common Stock	2	4	7	21
Municipal Bond	1	2	3	10
Preferred Stock	0	3	3	9

Let x_1 be the units of portfolio I, x_2 be the units of portfolio II, and x_3 be the units of portfolio III, then we can develop the following equations.

Common Stock: $2x_1 + 4x_2 + 7x_3 = 21$

Municipal Bond: $x_1 + 2x_2 + 3x_3 = 10$

Preferred Stock: $3x_2 + 3x_3 = 9$

Here,

$\Delta = 3$, $\Delta_1 = 9,\ \Delta_2 = 6,$ and $\Delta_3 = 3$

Now, according to Cramer's rule

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{9}{3} = 3,$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{6}{3} = 2,$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{3}{3} = 1$$

Exercise for Reader

1. Solve the following systems of linear equations.

   a) $6x_1 + 7x_2 = 56$

      $2x_1 + 3x_2 = 44$

   b) $2x_1 + 6x_2 = 22$

      $-x_1 + 5x_2 = 53$

   c) $2x_1 + 4x_2 - 3x_3 = 12$

      $3x_1 - 5x_2 + 2x_3 = 13$

      $-x_1 + 3x_2 + 2x_3 = 17$

   d) $5x_1 - 2x_2 + 3x_3 = 16$

      $2x_1 + 3x_2 - 5x_3 = 2$

      $4x_1 - 5x_2 + 6x_3 = 7$

2. Medication A is given every 4 hours and medication B is given twice per day, and the ratio of the dosage of A to the dosage of B is always 5 to 8.

   a) For patient I, the total intake of the two medications is 50.6 mg per day. Find the dosage of each administration of each medication for patient I.

   b) For patient II, the total intake of the two medications is 92 mg per day. Find the dosage of each administration of each medication for patient II.

3. Bob, a nutritionist who works for the University Medical Center, has been asked to prepare special diets for two patients, Susan and Tom. Bob has decided that Susan's meals should contain at least 400 mg of calcium, 20 mg of iron, and 50 mg of vitamin C, whereas Tom's meals should contain at least 350 mg of calcium, 15 mg of iron, and 40 mg of vitamin C. Bob has also decided that the meals are to be prepared from three basic foods: Food *A*, Food *B*, and Food *C*. The special nutritional contents of these foods are summarized in the accompanying table. Find how many ounces of each type of food should be used in a meal so that the minimum requirements of calcium, iron, and vitamin C are met for each patient's meals.

	Contents (mg/oz)		
	Calcium	Iron	Vitamin C
Food *A*	30	1	2
Food *B*	25	1	5
Food *C*	20	2	4