Lecture 33

Learning Objectives

At the end of this class, students should be able to:

- find the inverse of a matrix

The Inverse of a Square Matrix

Let A be a square matrix of size n. A square matrix A^{-1} of size n such that

$AA^{-1} = A^{-1}A = I$ is called the inverse of A.

A square matrix A is said to be singular if $|A| = 0$. A square matrix A is said to be non-singular if $|A| \neq 0$.

Not every square matrix has an inverse. A square matrix has an inverse if it is nonsingular.

We use the following formula to find an inverse

$$A^{-1} = \frac{Adj\ A}{|A|}, \text{ provided that } |A| \neq 0.$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

the matrix $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ is called the matrix of cofactors.

Where $A_{ij} = (-1)^{i+j} M_{ij}$

The transpose of this matrix of cofactors is called the adjoint of the matrix A and is denoted by Adj A.

Thus, $Adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Some Theorems on Inverse

We state some theorems concerning inverse of a matrix without proof.

1. The inverse of a matrix, if it exists, is unique.
2. The inverse of the product of two non-singular matrices is equal to the product of the inverses taken in the inverse order. In symbols, $(AB)^{-1} = B^{-1}A^{-1}$, where A and B are non-singular matrices.
3. If A is a non-singular matrix of order n, then $(A^{-1})' = (A')^{-1}$.

Illustration

For the matrix $A = \begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix}$ find A^{-1}, if it exists.

Solution

Here, $\begin{vmatrix} \mathbf{1} & \mathbf{5} \\ \mathbf{-2} & \mathbf{-1} \end{vmatrix} = \mathbf{-1 + 10 = 9 \neq 0}$

Since $|\boldsymbol{A}| \neq \mathbf{0}$ so $\boldsymbol{A}^{-1}$ exists

Now, cofactors are

$$\boldsymbol{A}_{11} = (-\mathbf{1})^{\mathbf{1+1}}(-\mathbf{1}) = -\mathbf{1}$$

$$\boldsymbol{A}_{12} = (-\mathbf{1})^{\mathbf{1+2}}(-\mathbf{2}) = \mathbf{2}$$

$$\boldsymbol{A}_{21} = (-\mathbf{1})^{\mathbf{2+1}}(\mathbf{5}) = -\mathbf{5}$$

$$\boldsymbol{A}_{22} = (-\mathbf{1})^{\mathbf{2+2}}(\mathbf{1}) = \mathbf{1}$$

Thus, the matrix of cofactors = $\begin{bmatrix} \mathbf{-1} & \mathbf{2} \\ \mathbf{-5} & \mathbf{1} \end{bmatrix}$ then

Adjoint (A) = $\begin{bmatrix} \mathbf{-1} & \mathbf{-5} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$

Now, using $\boldsymbol{A}^{-1} = \frac{\boldsymbol{Adjoint\ (A)}}{|\boldsymbol{A}|}$, we get

$$\boldsymbol{A}^{-1} = \frac{\mathbf{1}}{\mathbf{9}}\begin{bmatrix} \mathbf{-1} & \mathbf{-5} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{-1/9} & \mathbf{-5/9} \\ \mathbf{2/9} & \mathbf{1/9} \end{bmatrix}$$

Illustration

For the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 7 & 3 & -1 \end{bmatrix}$ find A^{-1}, if it exists.

Solution

Here, $|A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 7 & 3 & -1 \end{vmatrix} = 9 \neq 0$

Since $|A| \neq 0$ therefore A^{-1} exists.

Now, let us find cofactors

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 7 & -1 \end{vmatrix} = 9$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 7 & 3 \end{vmatrix} = 6$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 7 & -1 \end{vmatrix} = 6$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 7 & 3 \end{vmatrix} = 11$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

Thus, the matrix of cofactors = $\begin{bmatrix} -3 & 9 & 6 \\ -1 & 6 & 11 \\ 2 & -3 & -4 \end{bmatrix}$ then

Adjoint (A) = $\begin{bmatrix} -3 & -1 & 2 \\ 9 & 6 & -3 \\ 6 & 11 & -4 \end{bmatrix}$

Now $A^{-1} = \dfrac{Adj(A)}{|A|}$

$$= \frac{1}{9}\begin{bmatrix} -3 & -1 & 2 \\ 9 & 6 & -3 \\ 6 & 11 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & -1/9 & 2/9 \\ 1 & 2/3 & -1/3 \\ 2/3 & 11/9 & -4/9 \end{bmatrix}$$

Exercise for Reader

For the following matrices:

i) find $|A|$,

ii) does A^{-1} exist? explain why?

iii) find A^{-1} if it exists.

a) $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$

b) $A = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$

c) $A = \begin{bmatrix} 6 & -1 \\ -3 & 0 \end{bmatrix}$

d) $A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 7 \end{bmatrix}$

e) $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 0 \\ 1 & 9 & 6 \end{bmatrix}$

f) $A = \begin{bmatrix} 1 & 3 & 0 \\ 8 & 2 & 5 \\ 4 & 0 & 6 \end{bmatrix}$