Lecture 28

Learning Objectives
At the end of this class, students should be able to:

- understand the concept of differential equations and their solutions
- solve first order linear differential equations
- solve second order linear differential equations

Differential Equations
A differential equation is an equation that involves derivatives, or differentials. Differential equations are among the most useful tools for modeling continuous phenomena. Population dynamics, chemical kinetics, spread of disease, dynamic economic behavior, ecology, and the transmission of information are a few of the many areas in the physical, and social and life sciences that can be studied using differential equations.
For example,

$$\frac{dy}{dx} + 5x = 0, \frac{dP}{dt} = kP, \text{ and } \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = \sin x$$

are all differential equations.

Differential equations are divided into two classes: ordinary differential equations and partial differential equations. A differential equation involving derivatives with respect to a single variable is called an *ordinary differential equation*. A differential equation involving partial derivatives with respect to more than one independent variable is called a *partial differential equation*. Here, we shall deal with ordinary differential equations only.

Order and Degree of a Differential Equation
The order of the highest order derivative in a differential equation is called the order of the differential equation. The degree of the differential equation is the degree of the highest order derivative which occurs in it.
For example, the differential equation $x^2\left(\frac{dy}{dx}\right)^2 - y = 0$ is of order 1 and degree 2, whereas the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = \sin x$ is of second order and first degree.

Linear and Non-linear Differential Equations
An ordinary differential equation is divided into two classes, namely, linear equations and non-linear equations. A differential equation is called linear if (i) every dependent variable and every derivative involved occurs in the first degree only, and (ii) no product of dependent variables and/or derivatives occur. A differential equation which is not linear is called a non-linear differential equation.

For example, the equation $(\sin x)\frac{dy}{dx} + (\cos x)y = x^2 \sin x$ is a linear differential equation.

The equation $y\frac{dy}{dx} + (\cos x)y^3 = e^x$ is a nonlinear differential equation.

Solution of a Differential Equation

Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution or integral of the differential equation. It should be noted that a solution of a differential equation does not involve the derivatives of the dependent variable with respect to the independent variable or variables. For example, $y = ce^{2x}$ is a solution of $y' = 2y$ because by putting $y = ce^{2x}$ and $y' = 2ce^{2x}$, the given differential equation reduces to the identity $2ce^{2x} = 2ce^{2x}$.

General and Particular Solutions

If the number of arbitrary constants is equal to the order of the differential equation, it is called the general solution. A solution obtained by giving particular value to the arbitrary constant is called a particular solution.

Thus, $y = ce^{2x}$ is the general solution of the equation $y' = 2y$. But $y = 3e^{2x}$ is the particular solution (if $c = 3$). The particular solution is obtained by giving a particular value of this constant c.

Illustration

Find the general and particular solution for the differential equation: $\frac{dy}{dx} = 3x^2 + 7x + 5,\ f(0) = 5.$

Solution

Here, $\frac{dy}{dx} = 3x^2 + 7x + 5,$

Integrating, we get

$$y = \int (3x^2 + 7x + 5)dx$$

or, $$y = x^3 + \frac{7}{2}x^2 + 5x + c$$

This is the general solution of the given differential equation, where c is the constant of integration. According to question, $f(0) = 5$, i.e., when $x = 0$, $y = f(x) = 5$, substituting this, in the above general solution, we get

$$5 = (0)^3 + \frac{7}{2}(0)^2 + 5(0) + c$$

$$c = 5$$

Thus, the particular solution of the given differential equation is:

$$y = x^3 + \frac{7}{2}x^2 + 5x + 5.$$

Illustration

A particular drug was administered to a person in a dosage of 100 milligrams. After 6 hours, a blood sample reveals that the amount in the system is 40 milligrams. If V equals the amount of the drug in the bloodstream after t hours and V_o equals the amount in the bloodstream at $t = 0$, the decay occurs at a rate $\frac{dV}{dt} = -kV$. Find the function that can describe the drug level decay.

Solution

The given differential can be written as $\frac{dV}{V} = -k\ dt$

Integrating, $V = -kt + \ln c$

or $V = ce^{-kt}$

According to question, when $t = 0$, $V = 100$, thus we have

$$c = 100$$

Thus, the required solution of the differential equation is

$$V = 100e^{-kt}$$

According to the question, $V = 40$ when $t = 6$, with this substitution, we get

$$40 = 100e^{-k\times 6}$$

or

$$0.4 = e^{-k\times 6}$$

Taking natural logarithm on both sides of the equation, and then finding the value of k, we get

$$k = 0.1527$$

Thus, the particular function describing the drug level decay function is

$$V = 100e^{-0.1527t}$$

Illustration

Find the general and particular solution for the differential equation:

$$\frac{d^2y}{dx^2} = 6x + 18, f'(5) = -10, f(2) = 30.$$

Solution

Here, $\frac{d^2y}{dx^2} = 6x + 18$

Integrating, we get

$$\frac{dy}{dx} = 3x^2 + 18x + c_1 \qquad \ldots \text{(i)}$$

Again, integrating, we get

$$y = x^3 + 9x^2 + c_1x + c_2 \qquad \ldots \text{(ii)}$$

This is the general solution of the given differential equation, where $c_1 and\ c_2$ are constants of integration.

According to question, $f'(5) = -10$, i.e., when $x = 5$, $\frac{dy}{dx} = -10$, substituting this in the equation (i), we get

$$-10 = 3(5)^2 + 18(5) + c_1 \quad \Rightarrow c_1 = -175$$

Again, according to question, $f(2) = 30$, then from equation (ii), we get

$$30 = (2)^3 + 9(2)^2 + (-175)(2) + c_2 \quad \Rightarrow c_2 = 336$$

Thus, the required particular solution of the given differential equation is:

$$y = x^3 + 9x^2 - 175x + 336.$$

Exercise for Reader

1. Find the general and particular solution for the following differential equations:
   a) $\frac{dy}{dx} = 2x + 5,\ f(0) = 12$
   b) $\frac{dy}{dx} = 5,\ f(3) = 7$
   c) $\frac{dy}{dx} = 4(2x^2 + 3)^2,\ f(0) = 3$
   d) $\frac{dy}{dx} = x^2 + 3x + 8,\ f(1) = 7.5$
2. Assume that the rate of decay of a radioactive element in a bone is given by $\frac{dQ}{dt} = -0.00012Q$. If 60% of the original amount is found, estimate the age of the bone.
3. Find the general and particular solution for the following differential equations:
   a) $\frac{d^2y}{dx^2} = 6x - 9;\ f'(2) = 10,\ f(-2) = -10$
   b) $\frac{d^2y}{dx^2} = 25e^{5x};\ f'(0) = 4,\ f(0) = -2$
   c) $\frac{d^2y}{dx^2} = 6x + 8,\ f'(5) = -18,\ f(2) = 30$.