Lecture 27

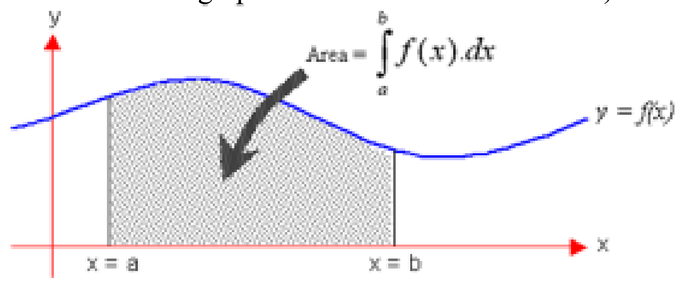
Learning Objectives

At the end of this class, students should be able to:

- find the area under the curve
- find the value of definite integral
- solve related problems

Area below the Graph of a Function

Integral calculus is primarily concerned with the area below the graph of a function (specifically, the area between the graph of a function and the x-axis).



If f is a continuous function on $[a, b]$ as given in the above figure and $f(x) \geq 0$ on $[a, b]$, then the area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is given by

$$\text{Shaded Area} = \int_a^b f(x)dx$$

Illustration

Let R be the region under the graph of $f(x) = (x+1)^2$ from $x = 1$ to $x = 3$. Find the area of the region R.

Solution

Required Area = $\int_1^3 (x+1)^2\, dx$

To identify the value of this integral, we have to understand the concept of definite integral.

The Definite Integral

Let $f(x)$ be a function that is continuous on the interval $[a, b]$. Then the definite integral of f on the interval $[a, b]$, denoted by $\int_a^b f(x)dx$.

The function $f(x)$ is called the integrand, and the numbers a and b are called the lower and upper limits of integration, respectively. The process of finding a definite integral is called definite integration.

Fundamental Theorem of Calculus

Let f be a continuous function on the closed interval $[a, b]$; then the definite integral of f exists on this interval, and

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.
Thus, we apply the Fundamental Theorem of Calculus by using the following two steps.

1. Integration of $f(x)$: $\int_a^b f(x)dx = F(x)\Big|_a^b$
2. Evaluation of $F(x)$: $F(x)\Big|_a^b = F(b) - F(a)$

Illustration

Evaluate $\int_1^3 (x+1)^2\, dx$.

Solution

$$
\begin{aligned}
\int_1^3 (x+1)^2\, dx &= \frac{(x+1)^3}{3}\Big|_1^3 \\
&= \frac{(3+1)^3}{3} - \frac{(1+1)^3}{3} \\
&= \frac{4^3}{3} - \frac{2^3}{3} \\
&= \frac{64-8}{3} \\
&= \frac{56}{3}
\end{aligned}
$$

Illustration

Evaluate $\int_1^4 \frac{2x+1}{x^2+x-1} dx$

Solution

Let $x^2 + x - 1 = u$ then
$(2x+1)dx = du$

When $x = 1, u = 1^2 + 1 - 1 = 1$ and when $x = 4, u = 4^2 + 4 - 1 = 19$.

Then the integral becomes

$$
\begin{aligned}
\int_0^1 \frac{2x+1}{x^2+x-1} dx &= \int_1^{19} \frac{1}{u}\, du \\
&= \ln u\Big|_1^{19} \\
&= \ln 19 - \ln 1 \\
&= \ln 19 - 0 \\
&= \ln 19
\end{aligned}
$$

Illustration

Evaluate $\int_0^1 x^2 e^x dx$

Solution

First solve it by removing the limits, so

$$
\begin{aligned}
\int x^2 e^x\, dx &= x^2 \int e^x\, dx - \int \left\{ \frac{d}{dx}(x^2) \int e^x dx \right\} dx \\
&= x^2 \times e^x - \int \left\{ 2x \times e^x \right\} dx \\
&= x^2 e^x - 2\int x e^x\, dx
\end{aligned}
$$

$$= x^2e^x - 2\left[x\int e^x\,dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\}dx\right]$$

$$= x^2e^x - 2\left[x \times e^x - \int\{1 \times e^x\}dx\right]$$

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right]$$

$$= x^2e^x - 2\left[xe^x - e^x\right] + c$$

$$= e^x(x^2 - 2x + 2) + c$$

Now

$$\int_0^1 x^2e^x dx = \{e^x(x^2 - 2x + 2) + c\}\Big|_0^1$$

$$= \left[e^1(1^2 - 2\times1 + 2) + c\right] - \left[e^0(0^2 - 2\times0 + 2) + c\right]$$

$$= e - 2$$

Illustration

The rate of change of temperature 1 hour after *x* a milligram of a drug is administered is $\frac{dT}{dx} = 3x - \frac{x^2}{4}$ $\;0 \le x \le 8$. What total change of temperature occurs when the dosage changes from 1 to 4 milligrams?

Solution

Here, $\frac{dT}{dx} = 3x - \frac{x^2}{4}$

The required change in the temperature when the dosage changes from 1 to 4 milligrams

$$= \int_1^4 \left(\frac{dT}{dx}\right) dx$$

$$= \int_1^4 \left(3x - \frac{x^2}{4}\right) dx$$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{12}\right)\Big|_1^4$$

$$= \left(\frac{3\times4^2}{2} - \frac{4^3}{12}\right) - \left(\frac{3\times1^2}{2} - \frac{1^3}{12}\right)$$

$$= \left(\frac{48}{2} - \frac{64}{12}\right) - \left(\frac{3}{2} - \frac{1}{12}\right)$$

$$= \frac{207}{12}$$

$$= 17\frac{3}{4}$$

Thus, the total change in temperature when the dosage changes from 1 to 4 milligrams = $17\frac{3}{4}$.

Average Value of a Continuous Function

The average value of a continuous function $y = f(x)$ over the interval [*a*, *b*] is given by

Average Value = $\frac{1}{b-a}\int_a^b f(x)\,dx$

Illustration

A drug manufacturer has developed a time-release capsule with the number of milligrams of the drug in the bloodstream given by $S = 30x^{18/7} - 240x^{11/7} + 480x^{4/7}$ where *x* is in hours

and $0 \le x \le 4$. Find the average number of milligrams of the drug in the bloodstream for the first 4 hours after a capsule is taken.
Solution

The average value of $S(x)$ over the interval $0 \le x \le 4$ is given by

$$\frac{1}{4-0}\int_0^4 (30x^{18/7} - 240x^{11/7} + 480x^{4/7})\,dx$$

$$= \frac{1}{4}\left(30 \times \frac{7}{25}x^{25/7} - 240 \times \frac{7}{18}x^{18/7} + 480 \times \frac{7}{11}x^{11/7}\right)\Big|_0^4$$

$$= \frac{1}{4}\left(30 \times \frac{7}{25} \times 4^{25/7} - 240 \times \frac{7}{18} \times 4^{18/7} + 480 \times \frac{7}{11} \times 4^{11/7}\right) - \frac{1}{4} \times 0$$

$$= \frac{1}{4}(1187.11 - 3297.54 + 2697.99) - 0$$

$$= \frac{1}{4}(587.56)$$

$$= 146.89$$

Thus, the average amount of the drug in the bloodstream for the first 4 hours after a capsule is taken = 146.89 milligrams

Improper Integrals

We have studied definite integrals in which the limits *a* and *b* on the integral $\int_a^b f(x)\,dx$ were real numbers and the integrand $f(x)$ was continuous. But there are certain integrals in which

- *a* or *b* or both are infinite, or
- $f(x)$ becomes infinite at some point in the interval (*a, b*).

Such integrals are called improper integrals.

Improper integrals are evaluated by applying the following techniques.

1. If $f(x)$ is continuous over the interval and the limit exists, then

   - $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$
   - $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$
   - $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

   where $-\infty < c < \infty$, provided that improper integral exists.

2. If $f(x) \to \infty$ as $x \to a$, then $\int_a^b f(x)\,dx = \lim_{h\to 0} \int_{a+h}^b f(x)\,dx$
3. If $f(x) \to \infty$ as $x \to b$, then $\int_a^b f(x)\,dx = \lim_{h\to 0} \int_a^{b-h} f(x)\,dx$

In each case, we say that the improper integral converges if the limit exists. If the limit fails to exist, the improper integral is said to be divergent.

Illustration

Evaluate $\int_2^\infty \frac{dx}{(1+x)^{3/2}}$ if it exists.

Solution

$$\lim_{t\to\infty}\int_2^t \frac{dx}{(1+x)^{3/2}} = \lim_{t\to\infty}\int_2^t (1+x)^{-3/2}dx$$

$$= \lim_{t\to\infty}\left[-2(1+x)^{-1/2}\Big|_2^t\right]$$

$$= \lim_{t\to\infty}\left[-2(1+t)^{-\frac{1}{2}} + \frac{2}{\sqrt{3}}\right]$$

$$= \lim_{t\to\infty}\left[-\frac{2}{\sqrt{1+t}} + \frac{2}{\sqrt{3}}\right]$$

$$= \left[-\frac{2}{\sqrt{1+\infty}} + \frac{2}{\sqrt{3}}\right]$$

$$= 0 + \frac{2}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}$$

Hence, $\int_2^\infty \frac{dx}{(1+x)^{3/2}} = \frac{2}{\sqrt{3}}$

Illustration

Determine if $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent. If convergent, find its value.

Solution

Here, $\frac{1}{\sqrt{x}} \to \infty \; as \; x \to 0.$

$$\therefore \int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{h\to 0}\int_h^1 \frac{1}{\sqrt{x}}dx$$

$$= \lim_{h\to 0}\left[2\sqrt{x}\Big|_h^1\right]$$

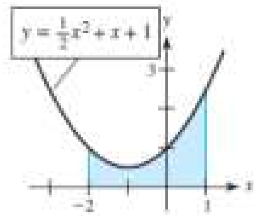
$$= \lim_{h\to 0}(2 - 2\sqrt{h})$$

= 2, which is a finite number.

Hence, the integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent, and the value of this integral is 2.

Exercise for Reader

1. For the following figure, write the integral that describes the area of the shaded region and find the area.



2. Evaluate the following integrals.

a) $\int_0^1 3x^2 e^{x^3}\, dx$ b) $\int_1^3 \frac{2x+3}{x^2+3x}\, dx$

c) $\int_1^e \frac{(\ln x)^2}{x}\, dx$ d) $\int_e^{e^2} \frac{1}{x \ln x}\, dx$

e) $\int_0^1 x e^x\, dx$ f) $\int_1^2 x^2 \ln x\, dx$

3. The rate of change in temperature, dT/dx, is $\frac{dT}{dx} = \frac{1}{20} x\sqrt{10 - x}$, $0 \le x \le 10$ where x is the number of milligrams given. What is the total temperature change when the dosage changes from 0 to 4 milligrams?

4. Suppose V represents the total flow in cubic units per second of blood through an artery whose radius is R, and suppose r is the distance of a particle of blood from the center of the artery. Assume that $\frac{dV}{dr} = 2\pi C(Rr - r^2)$ where C is a constant that depends on the units used.

   a) Compute $V = \int_R^{1.1R} 2\pi C(Rr - r^2)\, dr$ to find the increase in blood flow when the radius of artery is increased by 10%.

   b) Find the increase in blood flow when the radius of the artery is increased by 20%.

5. Evaluate the following integrals.

   a) $\int_1^{\infty} \frac{2x}{(1+x^2)^2}\, dx$ b) $\int_0^8 \frac{dx}{\sqrt{8-x}}$