**Lecture 26**

**Learning Objectives**

At the end of this class, students should be able to:

- evaluate the integral by using integration by parts
- evaluate the integral by using partial fractions
- solve related problems

**Integration by Parts**

Let $u$ and $v$ be two functions of $x$. then

$$\int u\,v\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$$

This is the integral of the product of two functions, and is known as Integration by Parts. We can express the result as under:

$\int u\,v\,dx$ = (1st function) × (Integral of 2nd function)
– Integral of {(derivative of 1st function) × (Integral of 2nd function)}

where, $u$: 1st function and $v$: 2nd function

How can we select $u$ and $v$ to make integration by parts work? As a general guideline, we do the following.

First, identify the types of functions occurring in the problem in the order: "L-I-A-T-E"
(There are always exceptions, but these are generally helpful.)

L: Logarithmic Function
I: Inverse Trigonometric Function
A: Algebraic Function
T: Trigonometric Function
E: Exponential Function

Second, choose $u$ to equal the function whose type occurs first on the list. Then $v$ equals the rest of the integrand so that $uv$ equals the original integrand.

Let us try to understand with the help of the following illustrations.

*Illustration*

Evaluate $\int x\ln x\,dx$

*Solution*

The integral contains a logarithmic function $(\ln x)$ and an algebraic function ($x$). Thus, let $u = \ln x$ and $v = x$. Then the given integral should be written as $\int(\ln x)x\,dx$.

Now

$$\int(\ln x)x\,dx = \ln x\int x\,dx - \int\left\{\frac{d}{dx}(\ln x)\int x dx\right\}dx$$

$$= \ln x \times \frac{x^2}{2} - \int \left\{ \frac{1}{x} \times \frac{x^2}{2} \right\} dx$$
$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$
$$= \frac{x^2}{2} \ln x - \frac{1}{2} \times \frac{x^2}{2} + c$$
$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

*Illustration*

Evaluate $\int xe^x \, dx$

*Solution*

The integral contains an algebraic ($x$) and exponential $(e^x)$. Thus, let $u = x$ and $v = e^x$.

Now

$$\int xe^x \, dx = x \int e^x \, dx - \int \left\{ \frac{d}{dx}(x) \int e^x dx \right\} dx$$
$$= x \times e^x - \int \left\{ 1 \times e^x \right\} dx$$
$$= xe^x - \int e^x \, dx$$
$$= xe^x - e^x + c$$
$$= e^x (x-1) + c$$

*Illustration*

Evaluate $\int x^2 \sin x \, dx$

*Solution*

Since $x^2$ is an algebraic function and $\sin x$ is a trigonometric function. Thus, let $u = x^2$ and $v = \sin x$.

Now,

$$\int x^2 \sin x \, dx = x^2 \int \sin x \, dx - \int \{ \tfrac{d}{dx}(x^2) \int \sin x \, dx \} \, dx$$
$$= x^2(-\cos x) - \int \{ 2x \times (-\cos x) \} dx$$
$$= -x^2 \cos x + 2 \int x \cos x \, dx$$
$$= -x^2 \cos x + 2[x \int \cos x \, dx - \int \left\{ \tfrac{d}{dx}(x) \int \cos x \, dx \right\} dx]$$
$$= -x^2 \cos x + 2[x \sin x - \int \sin x \, dx]$$
$$= -x^2 \cos x + 2[x \sin x + \cos x] + c$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + c$$

In this example, integration by parts technique was adopted two times to obtain an answer. We should not switch choices for *u* and *v* in successive applications.

*Illustration*

Evaluate $\int x\sqrt{x+5} \, dx$

*Solution*

Since both of these are algebraic functions, the LIATE Rule of Thumb is not helpful. In such cases, we take v as the most complicated portion of the integrand that can be "easily' integrated and u be that portion of the integrand whose derivative is a "simpler" function than u itself.

Thus, let $v = \sqrt{x+5}$ and $u = x$.

Now

$$\begin{aligned}
\int x\sqrt{x+5}\,dx &= x\int \sqrt{x+5}\,dx - \int\left\{\frac{d}{dx}(x)\int \sqrt{x+5}dx\right\}dx \\
&= x \times \frac{2}{3}(x+5)^{3/2} - \int\left\{1 \times \frac{2}{3}(x+5)^{3/2}\right\}dx \\
&= \frac{2x}{3}(x+5)^{3/2} - \frac{2}{3}\int (x+5)^{3/2}\,dx \\
&= \frac{2x}{3}(x+5)^{3/2} - \frac{2}{3} \times \frac{2}{5}(x+5)^{5/2} + c \\
&= \frac{2x}{3}(x+5)^{3/2} - \frac{4}{15}(x+5)^{5/2} + c
\end{aligned}$$

**Integration by Partial Fractions**

A rational function is a quotient of two polynomials. There are rational functions which cannot be integrated by the techniques presented earlier. When this occurs, one possibility is to break it into partial fractions, and then find the integrals of these fractions.

A group of fractions connected by the signs addition and subtraction is reduced to a simpler form, i.e., a single fraction whose denominator is L. C. M. of the given fractions. For example,

$$\frac{1}{x+1} - \frac{1}{x+2} = \frac{(x+2)-(x+1)}{(x+1)\,(x+2)} = \frac{1}{(x+1)\,(x+2)}$$

The reverse process of separating a fraction into group of simple fractions is known as *partial fractions*.

*Illustration*

Evaluate $\int \frac{1}{x^2+3x+2}\,dx$

*Solution*

Here the integrand $\frac{1}{x^2+3x+2}$ is a rational function, moreover, we cannot apply any of the techniques studied earlier. Therefore, we need to resolve the integrand into partial fractions. For this purpose, we have to find the factors of the denominator, i.e. $x^2 + 3x + 2 = (x + 1)\,(x + 2)$.

$$\therefore \quad \frac{1}{x^2+3x+2} = \frac{1}{(x+1)\,(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

Thus, $\int \frac{1}{x^2+3x+2}\,dx = \int\left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx = \log\,(x+1) - \log\,(x+2) + c$

When the numerator is of lower degree than the denominator. Here two cases arise the linear factors in the denominator may be a) repeated and b) non-repeated.

When we have non-repeated linear factors in the denominator, we write the given fraction in the following form and find the values of A and B.

$$\frac{p(x)}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

*Illustration*

Evaluate $\int \frac{7x}{x^2 - 3x + 2}\, dx$

*Solution*

The expression $\frac{7x}{x^2 - 3x + 2}$ is equivalent to $\frac{7x}{(x-1)(x-2)}$ so it can be written as:

$$\frac{7x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

or, $7x = A(x-2) + B(x-1)$ .

To find $A$, put $x = 1$, we get,

$7 \times (1) = A(1 - 2) + B(1 - 1)$, i.e., $A = -7$.

Similarly, to find $B$, put $x = 2$, we get,

$7 \times (2) = A(2-2) + B(2-1)$, i.e., $B = 14$.

$$\therefore \quad \frac{7x}{(x-1)\,(x-2)} = \frac{-7}{x-1} + \frac{14}{x-2}$$

Thus, $\int \frac{7x}{x^2 - 3x + 2}\, dx = \int \frac{-7}{x-1}\, dx + \int \frac{14}{x-2}\, dx$

$= -7 \ln(x-1) + 14 \ln(x-2) + c$

When some linear factors of the denominator of the given fraction are repeated, then

$$\frac{p(x)}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

**Exercise for Reader**

1. Evaluate the following integrals.

   a) $\int xe^{-x} dx$ b) $\int x^2 e^x dx$

   c) $\int (x + 1) e^{2x} dx$ d) $\int x^2 \sqrt{1 + x}\; dx$

   e) $\int x^2 \ln x\; dx$ f) $\int x \sin^2 x\; dx$

2. Evaluate the following integrals.

   a) $\int \frac{5x}{x^2+3x+2} dx$ b) $\int \frac{2x-3}{x^2-2x-15} dx$

3. The rate of change in temperature, dT/dx, is $\frac{dT}{dx} = \frac{1}{20} x\sqrt{10-x}$, $0 \le x \le 10$ where x is the number of milligrams given. What is the general expression for total temperature.