Lecture 25

Learning Objectives

At the end of this class, students should be able to:

- apply initial conditions
- use various techniques of integration
- solve related problems

Initial Conditions

We get constant of integration *c* when we find integral. This constant of integration c may be of interest in some applications. In such cases, we may specify a point that is a solution of the integral, thereby allowing us to solve for *c*. This point is called an initial condition.

Illustration

Find a function $f(x)$ such that $f'(x) = 3x - 2$where $f(x) = 2$ when *x* = -1.

Solution

The integral of $f'(x) = 3x - 2$ is

$$f(x) = \int (3x - 2)dx$$

$$= \frac{3}{2}x^2 - 2x + c$$

According to question, $f(x) = 2$ when *x* = -1, then

$$2 = \frac{3}{2}(-1)^2 - 2 \times (-1) + c$$

or $c = -3/2$

Therefore, the specific integral of $f'(x) = 3x - 2$ that satisfies the given initial condition is $f(x) = \frac{3}{2}x^2 - 2x - \frac{3}{2}$.

Illustration

After introducing a bactericide into a culture, a biologist gives the rate of change of the number of bacteria present as $\frac{dN}{dt} = 60 - 12t$. If N(0) = 1200, find N(t), and N(7). When the number of bacteria be zero.

Solution

Here, rate of change $\boldsymbol{\frac{dN}{dt} = 60 - 12t}$. Then

$$N(t) = \int (60 - 12t)dt$$

or, $N(t) = 60t - 6t^2 + c$

When t = 0, *N*(0) = 1200, then

$$1200 = 60 \times 0 - 6 \times 0 + c$$

or, $c = 1200$

Hence, the required number of bacteria at time t is given by

$$N(t) = 60t - 6t^2 + 1200$$

When t = 7,

$$N(7) = 60 \times 7 - 6 \times 7 + 1200$$

$$= 738.$$

The number of bacteria will be zero when N(t) = 0.

i.e., $60t - 6t^2 + 1200 = 0$

or, $t^2 - 10t - 200 = 0$

or, $(t + 10)(t - 20) = 0$
Thus, $t = -10, 20$
Neglecting negative value of t.
Thus, when t = 20, number of bacteria becomes zero.

Techniques of Integration
If the function to be integrated is in the standard form, we can easily integrate it by using the formulas. When the integrand is not in the standard form, we have to convert it in the standard form. The following techniques of integration are available to us.

- Method of simplification
- Method of substitution
- Method of integration by parts
- Method of partial fraction

Method of Simplification
If the integrand is the product or the quotient of two functions. In this case we try to express it as a sum of two or more terms and integrate. Let us try to understand with the help of the following illustrations.

Illustration
Evaluate: $\int \frac{x+3}{x-1} dx$
Solution
The integrand is the quotient of two functions $x + 3$ and $x - 1$. The integral can be written as
$\int \frac{x+3}{x-1} dx = \int \frac{x+4-1}{x-1} dx$ [While rewriting the numerator, we need to take care of the denominator]

$$= \int \left[\frac{x-1}{x-1} + \frac{4}{x-1}\right] dx$$
$$= \int \left[1 + \frac{4}{x-1}\right] dx$$
$$= \int dx + 4\int \frac{1}{x-1} dx$$
$$= x + 4\ln(x-1) + c$$

Note: Whenever numerator is of equal or higher degree than the denominator. We divide the numerator by denominator until a remainder is obtained which is of lower degree than the denominator,
i.e., $\frac{p(x)}{q(x)} = \text{Quotient} + \frac{\text{Remainder}}{q(x)}$,
where $p(x)$ and $q(x)$ are polynomials and the degree of p is greater than or equal to that of q.

Illustration
Evaluate $\int \frac{x^2+3x+5}{x+1} dx$.
Solution
Here, $\frac{x^2+3x+5}{x+1} = (x+2) + \frac{3}{x+1}$, then

$$\int \frac{x^2+3x+5}{x+1} dx = \int \left[(x+2) + \frac{3}{x+1}\right] dx$$
$$= \int (x+2)dx + 3\int \frac{1}{x+1} dx$$
$$= \left(\frac{x^2}{2} + 2x\right) + 3\ln(x+1) + c$$

If there is surd in the denominator, we first rationalize it and then integrate. Let us try to understand with the help of the following illustration.

Illustration

Evaluate: $\int \frac{1}{\sqrt{x+1}-\sqrt{x}} dx$

Solution

The integrand contains a surd in the denominator; therefore, first we rationalize the expression in the denominator.

$$\begin{aligned}\int \frac{1}{\sqrt{x+1}-\sqrt{x}} dx &= \int \frac{1}{\sqrt{x+1}-\sqrt{x}} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} dx \\ &= \int \frac{\sqrt{x+1}+\sqrt{x}}{x+1-x} dx \\ &= \int (\sqrt{x+1}+\sqrt{x}) dx \\ &= \frac{(x+1)^{3/2}}{3/2} + \frac{(x)^{3/2}}{3/2} + c \\ &= \frac{2}{3}\left((x+1)^{3/2} + x^{3/2}\right) + c\end{aligned}$$

Method of Substitution

Sometimes, the integrand is not in the appropriate form so that we cannot use the integration formulas directly. In this situation, we need to change the variable by suitable substitution. We follow the following procedure:

Step 1: Select a substitution that appears to simplify the integrand. In particular, try to select u so that du is a factor in the integrand.
Step 2: Express the integrand entirely in terms of u and du, completely eliminating the original variable.
Step 3: Evaluate the new integral if possible.
Step 4: Express the answer found in step 3 in terms of the original variable.

Illustration

Evaluate $\int (3x+5)^5 dx$.

Solution

Let $3x+5=u$ then

$d(3x+5)=du$

or $3dx = du$

$$\left[\because \text{ If } y = f(x) \text{ then } dy = \left\{\frac{d}{dx} f(x)\right\} dx\right]$$

or $dx = du/3$

Then the integral becomes

$$\begin{aligned}\int (3x+5)^5 dx &= \int u^5 (du/3) \\ &= \frac{1}{3}\int u^5 du \\ &= \frac{1}{2} \times \frac{1}{6} u^6 + c \\ &= \frac{1}{12}(3x+5)^6 + c \qquad [\because u = 3x+5]\end{aligned}$$

In general,

1. $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + k$; for $n \neq -1$, and

2. $\int \dfrac{1}{(ax+b)}\,dx = \dfrac{1}{a}\ln|(ax+b)| + k$

We can use these results as formulas.

Illustration

Evaluate $\int (5x+2)^{-3}\,dx$.

Solution

$$\int (5x+2)^{-3}\,dx = \frac{(5x+2)^{-2}}{5\times(-2)} + c$$

$$\left[\because \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + k\right]$$

$$= -\frac{1}{12}(5x+2)^{-2} + c$$

Illustration

Evaluate $\int \dfrac{1}{(2x+3)}\,dx$.

Solution

$$\int \frac{1}{(2x+3)}\,dx = \frac{1}{2}\ln(2x+3) + c$$

$$\left[\because \int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln|(ax+b)| + k\right]$$

Illustration

Evaluate $\int 2x(x^2+5)^6\,dx$.

Solution

Let $x^2+5 = u$ then

or $2xdx = du$

or $dx = \dfrac{du}{2x}$

Then the integral becomes

$$\int 2x(x^2+5)^6\,dx = \int 2x \times u^6 \times \frac{du}{2x}$$

$$= \int u^6 du$$

$$= \frac{1}{7}u^7 + c$$

$$= \frac{1}{7}(x^2+5)^7 + c$$

Illustration

Evaluate $\int \dfrac{(\ln x)^2}{3x}\,dx$.

Solution

Let $\ln x = u$ then

or $\frac{1}{x}dx = du$

or $dx = x\,du$

Then the integral becomes

$$\int \frac{(\ln x)^2}{3x}dx \quad = \int \frac{(u)^2}{3x} \times x\,du$$

$$= \frac{1}{3}\int u^2 du$$

$$= \frac{1}{3} \times \frac{1}{3}u^3 + c$$

$$= \frac{1}{9}(\ln x)^3 + c$$

Illustration

Evaluate $\int x^2 e^{x^3-1}dx$

Solution

Let $x^3 - 1 = u$ then

$3x^2 dx = du$

or $dx = \frac{du}{3x^2}$

Then the integral becomes

$$\int x^2 e^{x^3-1}dx \;= \int x^2 e^u \times \frac{du}{3x^2}$$

$$= \frac{1}{3}\int e^u du$$

$$= \frac{1}{3}e^u + c$$

$$= \frac{1}{3}e^{x^3-1} + c$$

Exercise for Reader

1. Evaluate the following integrals.

a) $\int \operatorname{cos}ec^2(2x+1)\,dx$

b) $\int (2x+5)^4\,dx$

c) $\int \frac{dx}{(3-5x)^3}$

d) $\int 5x^3(7x^4-5)^{5/3}\,dx$

e) $\int \frac{7x^2\,dx}{(3x^3+8)}$

f) $\int x^2\sin(x^3-1)\,dx$

g) $\int \frac{2\sec^2 x}{\sqrt{5+3\tan x}}\,dx$

h) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

i) $\int x\,e^{3-x^2}\,dx$

j) $\int \frac{2e^x\,dx}{\sqrt{e^x+4}}$

k) $\int \frac{2x\,dx}{\sqrt{x^2+5}}$

l) $\int \frac{\sqrt{\ln x}\,dx}{x}$

2. A package of frozen strawberries is taken from a freezer at -5^0 C into a room at 20^0 C. At time t the average temperature of the strawberries is increasing at the rate of $10e^{-0.4t}$ degrees Celsius per hour. Find the temperature of the strawberries at time t.
Hint: use the fact, when t = 0, temperature T = -5^0 C.
3. A flu epidemic hits a town. Let P(t) be the number of persons sick with flu at time t, where time is measured in days from the beginning of the epidemic and P(0) = 100. Suppose that after t days the flu is spreading at the rate of $120t-3t^2$ people per day. Find the formula for P(t).
4. A learning rate is given by $\frac{dL}{dt} = 0.06t - 0.0006t^2$, where L is the number of words learned and t is time in minutes. Find L(t), and L(30) if L = 0 when t = 0.