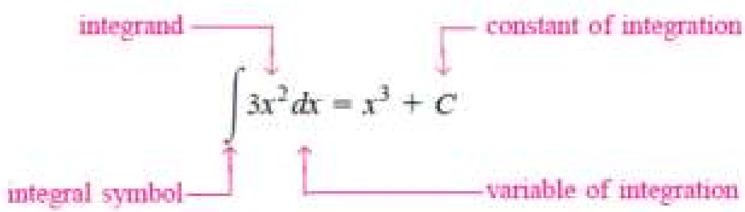
Lecture 24

Learning Objectives
At the end of this class, students should be able to:
- understand the concept of indefinite integral
- use integration rules
- solve related problems

Integration
The reverse process of differentiation is known as integration. We use the symbol $\int f(x)dx$ to represent the integration of $f(x)$. We write $\int f(x)\,dx = F(x) + c$ if $F'(x) = f(x)$. This integral is called indefinite integral because it involves a constant c that can take on any value. The symbol $\int$ is called an integral sign and the function $f(x)$ is called the integrand. The symbol dx indicates that the integration is performed with respect to the variable x. The letter c is called the constant of integration.
These features are displayed in the following diagram for the indefinite integral of $f(x) = 3x^2$:



Note: While finding integral, if $F'(x) = f(x)$, then the integration $\int f(x)dx = F(x) + c$ is correct, but if $F'(x)$ is anything other than $f(x)$, we must have committed a mistake.

Rules for Integrating Common Functions
- The constant rule: $\int k\,dx = kx + c$ for constant k
- The power rule: $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ for all $n \neq -1$
- Constant Multiple of a Function: $\int [k.f(x)]dx = k \int f(x)dx$ where k is a constant
- The logarithmic rule: $\int \dfrac{1}{x}dx = \ln|x| + c$ for all $x \neq 0$
- The exponential rule: $\int e^{kx}\,dx = \dfrac{e^{kx}}{k} + c$ for constant $k \neq 0$
- $\int \sin x\,dx = -\cos x + c$
- $\int \cos x\,dx = \sin x + c$
- $\int \sec^2 x\,dx = \tan x + c$
- $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
- $\int \sec x \tan x\,dx = \sec x$
- $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$

Illustration

Evaluate the following integrals:

a) $\int 5dx$

b) $\int 3x^4 dx$

Solution

a) $\int 5dx = 5x + c$ Check: $\frac{d}{dx}(5x + c) = 5$

b) $\int 3x^4 dx = \frac{3}{5}x^5 + c$ Check: $\frac{d}{dx}(\frac{3}{5}x^5 + c) = 3x^4$

Properties of Integration

1. A constant factor can be moved to the front of an indefinite integral:

$$\int [k.f(x)]dx = k\int f(x)dx$$

2. The integral of a sum or a difference is the sum or the difference of the integral:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Illustration

Evaluate the following integrals:

a) $\int (x^{1/3} - 3x^{-2/3} + 6)dx$

b) $\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right)dx$

c) $\int \left(3e^u + \frac{6}{u} + \ln 2\right)du$

Solution

a) $\int (x^{1/3} - 3x^{-2/3} + 6)dx$ $= \int x^{1/3}dx - 3\int x^{-2/3}dx + 6\int dx$

$= \frac{3}{4}x^{4/3} - 9x^{1/3} + 6x + c$

b) $\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right)dx$ $= \frac{1}{2}\int \frac{1}{x}dx - 2\int x^{-2}dx + 3\int x^{-1/2}dx$

$= \frac{1}{2}\ln x - \frac{2}{(-1)}x^{-1} + 6x^{1/2} + c$

$= \frac{1}{2}\ln x + \frac{2}{x} + 6x^{1/2} + c$

c) $\int \left(3e^u + \frac{6}{u} + \ln 2\right)du$ $= 3\int e^u du + 6\int \frac{1}{u}du + \ln 2\int du$

$= 3e^u + 6\ln u + u\ln 2 + c$

Illustration

Evaluate the following integrals:

a) $\int \frac{dx}{1-\cos 2x}$

b) $\int \sin 3x \cos 4x \, dx$

c) $\int 5e^{2x} dx$

Solution

a) $\int \frac{dx}{1-\cos 2x}$ $= \int \frac{dx}{1-(1-2sin^2 x)}$

$= \int \frac{dx}{2sin^2 x}$
$= \frac{1}{2}\int cosec^2\, x\, dx$
$= -\frac{1}{2}\cot x + c$

b) $\int \sin 3x \cos 4x\, dx$

$= \frac{1}{2}\int 2\sin 3x \cos 4x\, dx$
$= \frac{1}{2}\int [\sin(3x + 4x) + \sin(3x - 4x)]dx$
$= \frac{1}{2}\int (\sin 7x - \sin x)dx$
$= \frac{1}{2}\int \sin 7x\, dx - \frac{1}{2}\int \sin x\, dx$
$= \frac{1}{2}\left(\frac{-\cos 7x}{7} + \cos x\right) + c$

c) $\int 5e^{2x}dx$

$= 5\int e^{2x}dx$
$= 5 \times \frac{e^{2x}}{2} + c$
$= \frac{5e^{2x}}{2} + c$

Exercise for Reader

Evaluate the following integrals.

a) $\int 5dx$ b) $\int x^{1/2}dx$

c) $\int x^{-4/5}dx$ d) $\int (3x - 2)dx$

e) $\int (2 - 4x + 7x^2 - x^3)dx$ f) $\int \left(x - \frac{1}{x}\right)dx\; dx$

g) $\int \left(\frac{1}{x^2} + \frac{1}{2x} - x + 3x^{-1/2}\right)$ h) $\int \frac{e^{4x} + e^{3x} + 1}{e^{3x}}\, dx$

i) $\int sin^2 x\, dx$ j) $\int \sin 5x \cos 3x\;\; dx$