Lecture 23

Learning Objectives

At the end of this class, students should be able to:

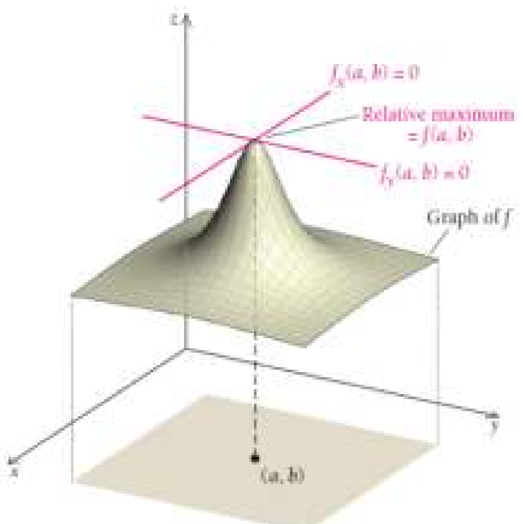
- find maximum and/or minimum value of the given functions

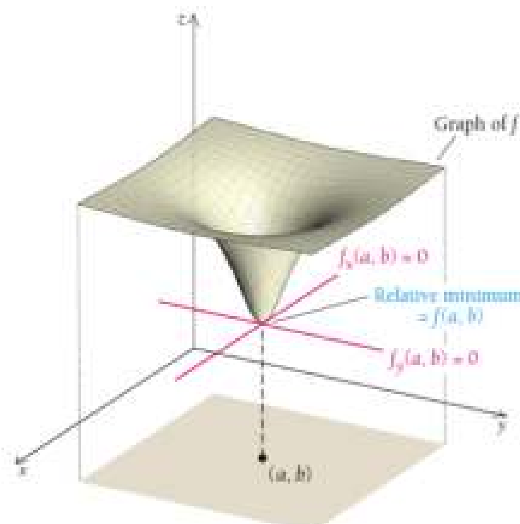
Maxima and Minima

A function $u = f(x, y)$ of two variables:

- has a relative maximum at (a, b) if $f(x, y) \le f(a, b)$ for all points (x, y) in a region containing (a, b).
- has a relative minimum at (a, b) if $f(x, y) \ge f(a, b)$ for all points (x, y) in a region containing (a, b).

The following figures represent the relative maximum and relative minimum.





Procedure for finding Maxima and Minima

We follow the following steps to find the relative maximum and minimum values of *f:*

1. Find f_x, f_y, f_{xx}, f_{xy}, and f_{yy},
2. Solve the system of equations $f_x = 0, f_y = 0$ to find the critical values. Let (a, b) represents a solution.
3. Find the value of D, where $D = f_{xx}.f_{yy} - (f_{xy})^2$. Evaluate the value of D at (a, b).
4. Then
   - $f(x, y)$ has a relative maximum at (a, b) if $D > 0$ and $f_{xx}(a,b) < 0$.
   - $f(x, y)$ has a relative minimum at (a, b) if $D > 0$ and $f_{xx}(a,b) > 0$.

- $f(x, y)$ has neither a maximum nor a minimum at (a, b) if $D < 0$. The function has a saddle point at (a, b).
- $f(x, y)$ has a maximum at (a, b) if $D > 0$ and $f_{xx}(a,b) < 0$.
- This test fails if D = 0 at (a, b).

Illustration
Find the relative maximum and minimum values of the function: $f(x, y) = x^2 + y^2 - 10x - 2y + 36$.
Solution
Here, $f(x, y) = x^2 + y^2 - 10x - 2y + 36$ then
$f_x = 2x - 10$ and
$f_y = 2y - 2$
For critical points, setting $f_x = 0$ and $f_y = 0$, we get
$2x - 10 = 0$ and $2y - 2 = 0$
Solving above equations, we get $x = 5$ and $y = 1$.
Now, $f_{xx} = 2, f_{xy} = 0,$ and $f_{yy} = 2$.
Here, $D = f_{xx}.f_{yy} - (f_{xy})^2 = 2 \times 2 - 0 = 2$
At $(x, y) = (5, 1)$,
$D = 2 > 0$
$f_{xx} = 2 > 0$
Thus, the given function has relative minimum at $(x, y) = (5, 1)$.
The minimum value of the function can be found by substituting $(x, y) = (5, 1)$ in the given function.
$f(5, 1) = (5)^2 + (1)^2 - 10 \times 5 - 2 \times 1 + 36 = 10.$

Illustration
Suppose that a pharmaceutical company has developed the yearly profit function $P(x, y) = -22x^2 + 22xy - 11y^2 + 110x - 44y - 23$ where x is the number of bottles (in thousands) of a particular drug (Adult) produced per year, y is the number of bottles (in thousands) of a same drug (pediatrics) produced per year, and P is profit (in thousands of dollars). How many of each type of bottles should be produced per year to realize a maximum profit? What is the maximum profit?
Solution
Here, $P(x, y) = -22x^2 + 22xy - 11y^2 + 110x - 44y - 23$ then
$P_x = -44x + 22y + 110$ and
$P_y = 22x - 22y - 44$
For critical points, setting $P_x = 0$ and $P_y = 0$, we get
$-44x + 22y + 110 = 0$ and $22x - 22y - 44 = 0$
Solving above equations, we get $x = 3$ and $y = 1$.
Now, $P_{xx} = -44, P_{xy} = 22,$ and $P_{yy} = -22$.
Here, $D = P_{xx}.P_{yy} - (P_{xy})^2 = (-44) \times (-22) - (22)^2 = 484$
At $(x, y) = (3, 1)$,

$D = 484 > 0$

$P_{xx} = -44 < 0$

Thus, the profit will be maximum at $(x, y) = (3, 1)$.

The maximum profit can be found by substituting $(x, y) = (5, 1)$ in the given profit function.

$P(3, 1) = -22(3)^2 + 22 \times 3 \times 1 - 11(1)^2 + 110 \times 3 - 44 \times 1 + 23 = 120$.

Thus, a maximum profit of $120,000 is obtained by producing and selling 3,000 bottles (adult) and 1,000 bottles (pediatrics) per year

Exercise for Reader

1. Find the relative maximum and minimum values.
   a) $f(x, y) = 5 - x^2 - y^2$
   b) $f(x, y) = x^3 + y^3 - 3xy$
2. A new food is designed to add weight to mature beef cattle. The weight in pounds is given by $W = 13xy(20 - x - 2y)$, where x is the number of units of the first ingredient and y is the number of units of the second ingredient. How many units of each ingredient will maximize the weight? What is the maximum weight?
3. Safe Shades produces two kinds of sunglasses; one kind sells for $17, and the other for $21. The total revenue in thousands of dollars from the sale of x thousand sunglasses at $17 each and y thousand at $21 each is given by $R(x,y) = 17x + 21y$. The company determines that the total cost, in thousands of dollars, of producing x thousand of the $17 sunglasses and y thousand of the $21 sunglasses is given by $C(x,y) = 4x^2 - 4xy + 2y^2 - 11x + 25y - 3$. Find the number of each type of sunglasses that must be produced and sold in order to maximize profit.
4. A firm produces two types of earphones per year: x thousand of type A and y thousand of type B. If the revenue and cost equations for the year are (in millions of dollars) $R(x,y) = 2x + 3y$ and $C(x,y) = x^2 - 2xy + 2y^2 + 6x - 9y + 5$ determine how many of each type of earphone should be produced per year to maximize profit. What is the maximum profit?
5. In a learning experiment, a subject is first given x minutes to examine a list of facts. The fact sheet is then taken away and the subject is allowed y minutes to prepare mentally for an exam based on the fact sheet. Suppose it is found that the score achieved by a particular subject is related to x and y by the formula $S = -x^2 + xy + 10x - y^2 + y + 15$
   a) What score does the subject achieve if he takes the test "cold" (with no study or contemplation)?
   b) How much time should the subject spend in study and contemplation to maximize his score? What is the maximum score?