Lecture 22

Learning Objectives
At the end of this class, students should be able to:
- find second order partial derivatives

Higher-Order Partial Derivatives
If $u = f(x, y)$ is a function of two variables, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of two variables and their partial derivatives can be taken. Hence, we can differentiate them with respect to x and y again and find,

$\frac{\partial^2 f}{\partial x^2}$, the derivative of f taken twice with respect to x,
$\frac{\partial^2 f}{\partial x \partial y}$, the derivative of f with respect to y and then with respect to x,
$\frac{\partial^2 f}{\partial y \partial x}$, the derivative of f with respect to x and then with respect to y,
$\frac{\partial^2 f}{\partial y^2}$, the derivative of f taken twice with respect to y.

Notations:

- $f_x = \frac{\partial f}{\partial x}$
- $f_y = \frac{\partial f}{\partial y}$
- $f_{xx} = \frac{\partial^2 f}{\partial x^2}$
- $f_{yy} = \frac{\partial^2 f}{\partial y^2}$
- $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$
- $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

Illustration
If $f(x, y) = x^2 + e^{xy}$ find the second order partial derivatives.
Solution
We have $f(x, y) = x^2 + e^{xy}$ then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + e^{xy}) = 2x + ye^{xy}$$

Again,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + ye^{xy})$$

or $$\frac{\partial^2 f}{\partial x^2} = 2 + y^2 e^{xy}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + ye^{xy})$$

or $$\frac{\partial^2 f}{\partial y \partial x} = 0 + y\frac{\partial}{\partial y}(e^{xy}) + e^{xy}\frac{\partial}{\partial y}(y)$$

$$= xye^{xy} + e^{xy}$$

Now,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + e^{xy}) = xe^{xy} \text{ then}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(xe^{xy})$$

or $$\frac{\partial^2 f}{\partial x \partial y} = x\frac{\partial}{\partial x}(e^{xy}) + e^{xy}\frac{\partial}{\partial x}(x)$$

$$= xye^{xy} + e^{xy}$$

Again,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(xe^{xy})$$

or $$\frac{\partial^2 f}{\partial y^2} = x^2 e^{xy}$$

Illustration

Find the second-order partial derivatives f_{xx}, f_{xy}, f_{yz} and f_{zz} of the function: $f(x,\ y,\ z) = x^2 y^2 + z^2$.

Solution

We have $f(x,\ y,\ z) = x^2 y^2 + z^2$ then

$$f_x = \frac{\partial}{\partial x}(x^2 y^2 + z^2) = 2xy^2,$$

$$f_{xx} = \frac{\partial}{\partial x}(2xy^2) = 2y^2,$$

$$f_{xy} = \frac{\partial}{\partial y}(2xy^2) = 4xy,$$

$$f_y = \frac{\partial}{\partial y}(x^2 y^2 + z^2) = 2x^2 y,$$

$$f_{yz} = \frac{\partial}{\partial z}(2x^2 y) = 0,$$

$$f_z = \frac{\partial}{\partial z}(x^2 y^2 + z^2) = 2z,$$

$$f_{zz} = \frac{\partial}{\partial z}(2z) = 2$$

Illustration

If $u = \ln(x^2+y^2)$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution

We have, $u = \ln(x^2+y^2)$ then

$$\frac{\partial u}{\partial x} = \frac{1}{(x^2+y^2)} \frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{(x^2+y^2)} \text{ and}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\times(2) - 2x\times 2x}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2}$$

$$= 0$$

Exercise for Reader

1. If $f(x, y) = e^{x^2+xy+y^2}$, show that $f_{xy} = f_{yx}$.
2. If $f(x, y) = xe^{xy}$, find the following: $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y^2}$.
3. If $f(x, y) = y^2 - \ln xy$, find the following: f_{xx}, f_{yy}, f_{xy}, and f_{yx}.
4. If $f(x, y, z) = x^2y^3z^4$, find the following: $f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}$, and f_{zy}.
5. If $u = \log\sqrt{(x^2+y^2+z^2)}$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{x^2+y^2+z^2}$.
6. If $u = e^{xyz}$, then prove that $f_{xyz} = (1+3xyz+x^2y^2z^2)e^{xyz}$.
7. If $u = x^2y + y^2z + z^2x$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2$.