Lecture 21

Learning Objectives

At the end of this class, students should be able to:

- find first order partial derivatives
- apply partial derivatives to find rate measures

Rules of Partial Differentiation

Quotient Rule

If $u = f(x, y) = g(x, y)/h(x, y)$ *and* $h(x, y) \neq 0$, where g and h are differentiable functions, then

$$\frac{\partial u}{\partial x} = \frac{h\frac{\partial g}{\partial x} - g\frac{\partial h}{\partial x}}{h^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{h\frac{\partial g}{\partial y} - g\frac{\partial h}{\partial y}}{h^2}$$

Let us try to understand this rule with the help of the following illustration.

Illustration

Let $f(x, y) = \frac{2x}{x^2 + y^2}$, find f_x and f_y.

Solution

Here, $f(x, y) = \frac{2x}{x^2 + y^2}$ then

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(\frac{2x}{x^2 + y^2}\right) \\ &= \frac{(x^2 + y^2) \times \frac{\partial}{\partial x}(2x) - 2x \times \frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{(x^2 + y^2) \times 2 - 2x \times 2x}{(x^2 + y^2)^2} \\ &= \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2} \\ &= \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2} \end{aligned}$$

Again,

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2}\right) \\ &= \frac{(x^2 + y^2) \times \frac{\partial}{\partial y}(2x) - 2x \times \frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{(x^2 + y^2) \times 0 - 2x \times 2y}{(x^2 + y^2)^2} \\ &= -\frac{4xy}{(x^2 + y^2)^2} \end{aligned}$$

General Power Function Rule

If $u = [f(x, y)]^n$, then

$$\frac{\partial u}{\partial x} = n[f(x, y)]^{n-1}.\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = n[f(x, y)]^{n-1}.\frac{\partial f}{\partial y}$$

Let us try to understand this rule with the help of the following illustration.

Illustration

Let $f(x, y) = (x^2 + 2y^3)^5$, find f_x and f_y .

Solution

Here, $f(x, y) = (x^2 + 2y^3)^5$ then

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}(x^2 + 2y^3)^5 \\ &= 5(x^2 + 2y^3)^4 \frac{\partial}{\partial x}(x^2 + 2y^3) \\ &= 5(x^2 + 2y^3)^4 \times 2x \\ &= 10x(x^2 + 2y^3)^4 \end{aligned}$$

Again,

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}(x^2 + 2y^3)^5 \\ &= 5(x^2 + 2y^3)^4 \frac{\partial}{\partial y}(x^2 + 2y^3) \\ &= 5(x^2 + 2y^3)^4 \times 6y^2 \\ &= 30y^2(x^2 + 2y^3)^4 \end{aligned}$$

Illustration

If $f(x, y, z) = 2x\sqrt{yz - 1} + x^2 z^3$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

Solution

We have $f(x, y, z) = 2x\sqrt{yz - 1} + x^2 z^3$ then

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(2x\sqrt{yz - 1} + x^2 z^3\right) \\ &= (\sqrt{yz - 1})\frac{\partial}{\partial x}(2x) + z^3 \frac{\partial}{\partial x}(x^2) \\ &= (\sqrt{yz - 1}) \times 2 + z^3 \times 2x \\ &= 2\sqrt{yz - 1} + 2xz^3 \end{aligned}$$

Again,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(2x\sqrt{yz - 1} + x^2 z^3\right) \\ &= 2x\frac{\partial}{\partial y}(yz - 1)^{1/2} + \frac{\partial}{\partial y}(x^2 z^3) \\ &= 2x \times \frac{1}{2}(yz - 1)^{-1/2} \frac{\partial}{\partial y}(yz - 1) + 0 \\ &= x(yz - 1)^{-1/2} \times z \\ &= \frac{xz}{\sqrt{yz - 1}} \end{aligned}$$

Similarly,

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(2x\sqrt{yz-1} + x^2z^3\right)$$

$$= 2x\frac{\partial}{\partial z}(yz-1)^{1/2} + x^2\frac{\partial}{\partial z}(z^3)$$

$$= 2x \times \frac{1}{2}(yz-1)^{-1/2}\frac{\partial}{\partial z}(yz-1) + x^2 \times 3z^2$$

$$= x(yz-1)^{-1/2} \times y + 3x^2z^2$$

$$= \frac{xy}{\sqrt{yz-1}} + 3x^2z^2$$

Illustration

Lincolnville Sporting Goods has the production function $p(x,y) = 2400x^{2/5}y^{3/5}$ for a certain product: where *p* is the number of units produced with *x* units of labor and *y* units of capital.

a) Find the number of units produced with 32 units of labor and 1024 units of capital.
b) Find the marginal productivities.
c) Evaluate the marginal productivities at *x* = 32 and *y* = 1024.
d) Interpret the meanings of the marginal productivities found in part (c).

Solution

We have $p(x,y) = 2400x^{2/5}y^{3/5}$

a) $p(32,\ 1024) = 2400 \times (32)^{2/5} \times (1024)^{3/5}$
$= 2400 \times 4 \times 64 = 614{,}400$

b) The marginal productivity of labor is:

$$\frac{\partial p}{\partial x} = 2400y^{3/5}\frac{\partial}{\partial x}(x^{2/5}) = 960\frac{y^{3/5}}{x^{3/5}}$$

The marginal productivity of capital is:

$$\frac{\partial p}{\partial y} = 2400x^{2/5}\frac{\partial}{\partial y}(y^{3/5}) = 1440\frac{x^{2/5}}{y^{2/5}}$$

c) When x = 32 and y = 1024,

Marginal productivity of labor = $\left.\frac{\partial p}{\partial x}\right|_{(32,\ 1024)}$

$$= 960\frac{(1024)^{3/5}}{(32)^{3/5}} = \frac{960 \times 64}{8} = 7{,}680$$

Marginal productivity of capital = $\left.\frac{\partial p}{\partial y}\right|_{(32,\ 1024)}$

$$= 1440\frac{(32)^{2/5}}{(1024)^{2/5}} = \frac{1440 \times 4}{16} = 360$$

d) Suppose that capital is fixed at 1024 units. Then the marginal productivity of labor tells us that a one-unit change in labor, from 32 to 33, will cause production to increase by about 7,680 units.

Similarly, suppose that labor is fixed at 32 units. Then the marginal productivity of capital tells us that a one-unit change in capital, from 1024 to 1025, will cause production to increase by about 360 units.

Exercise for Reader

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions:
   a) $f(x, y) = \frac{2x - 3y}{x^2 + y^2}$
   b) $f(x, y) = (3x + 5y^2)^{7/3}$
   c) $f(x, y) = 100e^{xy}$
   d) $f(x, y) = \ln(xy + 1)$
2. If $u = xe^{x-y} + ye^{y-x}$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = e^{x-y} + e^{y-x}$.
3. If $f(x, y, z) = e^{-x}\sqrt{3y + z}$, find the following: f_x, f_y, and f_z.
4. Suppose that the number of thousands of insects killed by two brands of pesticide is given by $f(x, y) = 10{,}000 - 6500e^{-0.01x} - 3500e^{-0.02y}$ where x is the number of liters of brand 1 and y is the number of liters of brand 2. What is the rate of change of insect deaths with respect to the number of liters of brand 1 if 100 liters of each brand are currently being used? What does this mean?