Lecture 20

Learning Objectives

At the end of this class, students should be able to:

- understand the concept of functions of several variables
- find first order partial derivatives

20.1 Functions of Several Variables

A company manufactures 10-speed and 3-speed bicycles. The weekly demand and cost equations are $p = 230 - 9x + y,\ q = 130 + x - 4y$ and $C(x, y) = 200 + 80x + 30y$ where \$*p* is the price of a 10-speed bicycle, \$*q* is the price of a 3-speed bicycle, *x* is the weekly demand for 10-speed bicycles, *y* is the weekly demand for 3-speed bicycles, and *C*(*x*, *y*) is the cost function. Find the weekly revenue function *R*(*x*, *y*) and the weekly profit function *P*(*x*, *y*).

Here *p* and *q* both depend upon two different independent variables *x* and *y*.

We know that,

Revenue = price × quantity

Here, revenue from 10-speed bicycles is:

$$R_1 = (230 - 9x + y)x = -9x^2 + xy + 230x$$

Revenue from 3-speed bicycles is:

$$R_2 = (130 + x - 4y)y = -4y^2 + xy + 130y$$

The weekly revenue function is:

$$\begin{aligned} R(x, y) &= -9x^2 + xy + 230x - 4y^2 + xy + 130y \\ &= -9x^2 - 4y^2 + 360x + 130y \end{aligned}$$

Thus, the weekly profit function is:

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= -9x^2 - 4y^2 + 360x + 130y - (200 + 80x + 30y) \\ &= -9x^2 - 4y^2 + 280x + 100y - 200 \end{aligned}$$

Here, both revenue function and profit function are functions of two variables.

In general, an equation of the form $u = f(x, y)$ describes a function of two independent variables if, for each permissible ordered pair (*x*, *y*), there is one and only one value of *u* determined by $f(x, y)$. The variables *x* and *y* are independent variables, and the variable *u* is a dependent variable.

The set of all ordered pairs of permissible values of *x* and *y* is the domain of the function, and the set of all corresponding values $f(x, y)$ is the range of the function. Similarly, $u = f(x, y, z)$ is the function of three variable.

Many formulas in mathematics can be considered as functions of two or more variables. We know that the area of a rectangle $A(x, y) = xy$, where *x* is the length and *y* is the breadth of the rectangle. Similarly, the volume of a box is given by $V(x, y, z) = xyz$, where x is the length, y is the breadth and z is the height of the box.

Illustration

For $f(x,y) = \frac{35}{x^2+3y}$ find the value of $f(4,-1)$ and $f(-3,3)$.

Solution

We have $f(x,y) = \frac{35}{x^2+3y}$

Now, $f(4,-1) = \frac{35}{4^2+3\times(-1)} = \frac{35}{13}$, and

$$f(-3,3) = \frac{35}{(-3)^2+3\times 3} = \frac{35}{18}$$

20.2 Partial Derivatives

Let $u = f(x,y)$ then we find the partial derivative of u with respect to x (denoted by $\frac{\partial u}{\partial x}$) by treating the variable y as a constant and taking the derivative of $u = f(x,y)$ with respect to x. The other notations to represent partial derivative of $u = f(x,y)$ with respect to x are $\frac{\partial f}{\partial x}$, $\frac{\partial}{\partial x} f(x,y)$, f_x, and u_x.

Similarly, we can also take the partial derivative of u with respect to y by holding the variable x constant and taking the derivative of $u = f(x,y)$ with respect to y. We denote this derivative as $\frac{\partial u}{\partial y}$.

Note that $\frac{du}{dx}$ represents the derivative of a function of one variable $u = f(x)$ whereas, $\frac{\partial u}{\partial x}$ represents the partial derivative of a function of two or more variables.

20.3 Rules of Partial Differentiation

The rules of partial differentiation follow exactly the same logic as simple differentiation. The only difference is that we have to decide how to treat the other variable.

Sum or Difference Rule

If $u = f(x, y) = g(x, y) \pm h(x, y)$, where g and h are differentiable functions, then

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial x} \pm \frac{\partial h}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial g}{\partial y} \pm \frac{\partial h}{\partial y}$$

Let us try to understand this rule with the help of the following illustration.

Illustration

Find the partial derivatives of the function $R(x, y) = -9x^2 + 30y^2 + 81xy$ with respect to x and y.

Solution

Here, $R(x, y) = -9x^2 + 30y^2 + 81xy$ then

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(-9x^2 + 30y^2 + 81xy)$$

$$= \frac{\partial}{\partial x}(-9x^2) + \frac{\partial}{\partial x}(30y^2) + \frac{\partial}{\partial x}(81xy)$$
$$= -18x + 0 + 81y$$
$$= -18x + 81y$$

Again

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(-9x^2 + 30y^2 + 81xy)$$
$$= \frac{\partial}{\partial y}(-9x^2) + \frac{\partial}{\partial y}(30y^2) + \frac{\partial}{\partial y}(81xy)$$
$$= 0 + 60y + 81x$$
$$= 60y + 81x$$

Illustration

Find the partial derivative of $f(x, y) = 5x^2 - 4xy + y^2$ with respect to x at the point (1, 2, - 2).

Solution

We have $f(x, y) = 5x^2 - 4xy + y^2$ then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(5x^2 - 4xy + y^2) = 10x - 4y$$

At the point (1, 2, - 2),

$$\frac{\partial f}{\partial x} = 10 \times 1 - 4 \times 2 = 2$$

Product Rule

If $u = f(x, y) = g(x, y).h(x, y)$, where g and h are differentiable functions, then

$$\frac{\partial u}{\partial x} = g\frac{\partial h}{\partial x} + h\frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = g\frac{\partial h}{\partial y} + h\frac{\partial g}{\partial y}$$

Let us try to understand this rule with the help of the following illustration.

Illustration

If $u = (3x + 2y)(5 - 2x)$ then find partial derivatives with respect to x and y.

Solution

Here, $u = (3x + 2y)(5 - 2x)$ then

$$\frac{\partial u}{\partial x} = (3x + 2y)\frac{\partial}{\partial x}(5 - 2x) + (5 - 2x)\frac{\partial}{\partial x}(3x + 2y)$$
$$= (3x + 2y)\ (-2) + (5 - 2x)\ (3)$$
$$= -12x - 4y + 15$$
$$\frac{\partial u}{\partial y} = (3x + 2y)\frac{\partial}{\partial y}(5 - 2x) + (5 - 2x)\frac{\partial}{\partial y}(3x + 2y)$$
$$= (3x + 2y)\ (0) + (5 - 2x)\ (2)$$
$$= -4x + 10$$

Exercise for Reader

1. Find the indicated value of the given function
   a) $P(8,3)$ for $P(n,r) = \frac{n!}{(n-r)!}$
   b) $V(2,5)$ for $V(r, h) = \pi r^2 h$
2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions:
   a) $f(x, y) = 3x^2 y^3$
   b) $f(x, y) = xy^3 + y$
   c) $f(x, y) = x^4 + 5x^2 y + 6y^2$
   d) $f(x, y) = (2x + 5y)(x^2 - 7xy + y^3)$