**Lecture 19**

**Learning Objectives**

At the end of this class, students should be able to:

- find the rate of change
- find higher order derivatives
- find maximum or minimum values of the given functions

## 19.1 The Derivative as a Rate of Change

An important interpretation of the derivative of a function at a point is as a rate of change. The derivative $f'(a)$ measures the rate of change of $f(x)$ at $x = a$.

*Illustration*

The highest recorded temperature in the state of Alaska was 100°F and occurred on June 27, 1915, at Fort Yukon. The *heat index* is the apparent temperature of the air at a given temperature and humidity level. If $x$ denotes the relative humidity (in percent), then the heat index (in degrees Fahrenheit) for an air temperature of 100°F can be approximated by the function $f(x) = 0.009x^2 + 0.139x + 91.875$

a) At what rate is the heat index changing when the humidity is 50%?

b) Write a sentence that explains the meaning of your answer in part (a).

*Solution*

a) We have $f(x) = 0.009x^2 + 0.139x + 91.875$, then

$$f'(x) = 0.018x + 0.139$$

When x = 50, then

$$f'(50) = 0.018 \times 50 + 0.139 = 1.039$$

b) If humidity changes by 1%, the heat index will change by about 1.039°F.

*Illustration*

The value of particular asset is estimated by the function $V = 240{,}000e^{-0.04t}$ where $V$ is the value of the asset and $t$ is the age of the asset, measured in years.

a) What is the value of the asset expected to equal when 5 years old?

b) Determine the general expression for the rate of change (instantaneous rate of change) in the value of the asset.

c) What is the rate of change expected to equal when the asset is 10 years old?

*Solution*

a) We have, $V = 240{,}000e^{-0.04t}$

When $t = 5$, $V = 240{,}000e^{-0.04\times5} = 240{,}00e^{-0.20} = 196{,}495.38$

After 5 years, the expected value of the asset will be $196,495.38.

b) Instantaneous rate of change in the value of the asset is given by

$$\frac{dV}{dt} = 240{,}000\frac{d}{dt}(e^{-0.04t})$$

$$= -9600e^{-0.04t}$$

Thus, the required expression for instantaneous rate of change in the value of the asset is

$$\frac{dV}{dt} = -9600e^{-0.04t}$$

c) When $t = 10$,

$$\frac{dV}{dt} = -9600e^{-0.04 \times 10}$$

$= -6,435.07$

Thus, the estimated rate of decrease in the value of the asset at the end of $10^{th}$ year is \$6,435.07 per year.

Note: *The negative sign indicates the decrease in value and positive sign indicates increase in value.*

*Illustration*

The growth of certain insects varies with temperature. Suppose a particular species of insect grows in such a way that the volume of an individual is $V(T) = 0.41(-0.01T^2 + 0.4T + 3.52)\ cm^3$, when the temperature is $T$°C, and that its mass is $m$ grams, where

$$m(V) = \frac{0.39V}{1 + 0.09V}$$

a) Find the rate of change of the insect's volume with respect to temperature.
b) Find the rate of change of the insect's mass with respect to volume.
c) When $T = 10$°C, what is the insect's volume? At what rate is the insect's mass changing with respect to temperature when $T = 10$°C?

*Solution*

a) We have $V(T) = 0.41(-0.01T^2 + 0.4T + 3.52)$, then the rate of change of the insect's volume with respect to temperature is given by

$$V'(T) = 0.41(-0.02T + 0.4)$$

b) We have $m(V) = \frac{0.39V}{1+0.09V}$, then the rate of change of the insect's mass with respect to volume is given by

$$m'(V) = \frac{(1+0.09V)\times 0.39 - 0.39V\times 0.09}{(1+0.09V)^2}$$

or, $$m'(V) = \frac{0.39}{(1+0.09V)^2}$$

c) When $T = 10$°C, insect's volume is given by

$$V(10) = 0.41(-0.01 \times 10^2 + 0.4 \times 10 + 3.52) = 2.6732\ cm^3$$

The rate of change of the insect's mass with respect to volume when $T = 10$°C is given by

$$\frac{dm}{dT} = \frac{dm}{dV} \times \frac{dV}{dT}$$

When $T = 10$°C, $V(10) = 2.6732\ cm^3$, then

$$\frac{dV}{dT} = V'(10) = 0.41(-0.02 \times 10 + 10) = 0.082$$ , and

$$\frac{dm}{dV} = m'(2.6732) = \frac{0.39}{(1+0.09\times 2.6732)^2} = 0.2534$$

Thus, $$\frac{dm}{dT} = \frac{dm}{dV} \times \frac{dV}{dT} = 0.2534 \times 0.082 = 0.020778 \text{ gm/°C}$$

### 19.2 Higher Order Derivatives

Let $y = f(x)$ be a differentiable function. Then

$y' = \frac{d}{dx}\{f(x)\}$ is called the first order derivative of $y = f(x)$.

If we differentiate it again, we get

$y'' = \frac{d}{dx}(y')$ which is called the second order derivative of $y = f(x)$.

Similarly,

$y''' = \frac{d}{dx}(y'')$ is the called the third order derivative of $y = f(x)$.

In general, the notation for the $n^{th}$ order derivative is $y^{(n)}\ or\ f^n(x) or\ \frac{d^ny}{dx^n}$.

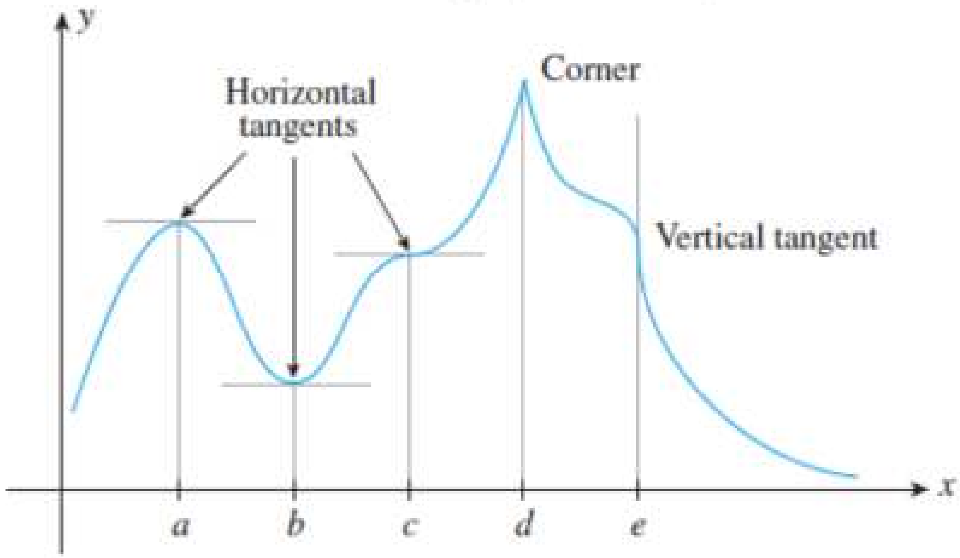
For example, if $y = x^5 + 3x^3 - 7x^2 + 10$ then

$y' = 5x^4 + 9x^2 - 14x$ is the first derivative of the given function.

$y'' = 20x^3 + 18x - 14$ is the second derivative of the given function. Similarly,
$y''' = 60x^2 + 18$ is the third derivative of the given function and so on.

### 19.3 Critical Values

A critical value of a function *f* is any number *x* in the domain of *f* such that $f'(x) = 0$ or $f'(x)$ does not exist. Let us consider the following graph of a function *f* (*x*).




In the above figure, the graph of a function has critical values at *a*, *b*, *c*, *d*, and *e*. We observe that $f'(x) = 0$ at *a*, *b*, and *c*. $f'(x)$ is not defined at *d* because the function has a corner at that point. Similarly, $f'(x)$ does not exist at *e* because the tangent line is vertical at that point.

A 'peak' on the graph of a function *f* is known as a relative maximum of *f*, and a 'valley' is a relative minimum. Thus, a relative maximum is a point on the graph of *f* that is at least as high as any nearby point on the graph, while a relative minimum is at least as low as any nearby point.

In the above figure, relative maximum occurs at *a* and *d* whereas relative minimum occurs at *b*. We have to apply standard test to identify whether the function has relative maximum or minimum at the particular critical point.

### 19.4 The Second Derivative Test

The following steps will be followed for identifying optimum values of the function.

1. Find all critical points $x^*$, such that $f'(x^*) = 0$.
2. For any critical point $x^*$, determine the value of $f''(x^*)$.
   a) If $f''(x^*) > 0$, the given function has a relative minimum at $x^*$.
   b) If $f''(x^*) < 0$, the given function has a relative maximum at $x^*$.
   c) If $f''(x^*) = 0$, no conclusion can be drawn. Another test is required.

*Illustration*

Find the local maxima and minima for function $f(x) = x^3 - 6x^2 + 9x + 5$.

*Solution*

We have $f(x) = x^3 - 6x^2 + 9x + 5$ then

$$f'(x) = 3x^2 - 12x + 9$$

For critical value, setting $f'(x) = 0$

i.e. $3x^2 - 12x + 9 = 0$

or $3(x-1)(x-3) = 0$

Thus, the critical values are $x = 1$ and $x = 3$.

Now, $f''(x) = 6x - 12$

When $x = 1$,

$$f''(1) = 6\times 1 - 12 = -6 < 0$$

Thus, the given function has a relative maximum at $x = 1$.

When $x = 3$,

$$f''(3) = 6\times 3 - 12 = 6 > 0$$

Thus, the given function has a relative minimum at $x = 3$.

*Illustration*

A rectangular garden of area 75 square feet is to be surrounded on three sides by a brick wall costing \$10 per foot and on one side by a fence costing \$5 per foot. Find the dimensions of the garden such that the cost of material is minimized.

*Solution*

Let x feet and y feet be the two sides of the rectangular garden.

Then, according to the question, the area of the garden xy = 75 …(1)

Now, the fencing cost of three sides by brick wall is given by 10(2x + y).

Similarly, fencing cost of the remaining one side is given by 5y.

Therefore, the total fencing cost will be 10(2x + y) + 5y = 20x + 15y = T (say)

Now, we have to find the values of x and y for minimum T.

Using equation (1), we can write $T = \frac{1500}{y} + 15y$

Then $T' = -\frac{1500}{y^2} + 15$

For stationary points, $T' = 0$

i.e., $-\frac{1500}{y^2} + 15 = 0$

or, $y^2 = 100$

or, $y = \pm 10$

The value of y cannot be negative so we neglect $y = -10$.

Now, $T'' = \frac{3000}{y^2}$

When $y = 10$, $T'' = \frac{3000}{10^2} > 0$

Thus, the total fencing cost T is minimum for $y = 10$.

From equation (1), $x = 7.5$ and the minimum cost

$$T = \frac{1500}{10} + 15 \times 10 = 300$$

Thus, the required dimensions of the garden are 7.5 feet and 10 feet, and the minimum cost of the materials is \$300.

**Exercise for Reader**

1. Liquid is pouring into a large vat. After t hours, there are $5t - \sqrt{t}$ gallons in the vat. At what rate is liquid flowing into the vat (in gallons per hour) when t = 4?
2. An epidemic is spreading through a large western region. Health official estimates that the number of persons who will be affected by the disease is a function of time since the disease was first detected. Specifically, the function is $n = f(t) = 300t^3 - 20t^2$ where n equals the number of persons and $0 \le t \le 60$, measured in days.
   a) How many persons are expected to have caught the disease after 20 days?
   b) At what rate is the disease spreading at $t = 30$ ?
3. The population of a rare species of wild life is declining. The function $P = 75{,}000e^{-0.025t}$ estimates the population P of the species as a function of time, measured in years since 1980. Determine the general expression for the instantaneous rate of change in the population. At what rate is the population estimated is declining in the year 1996?
4. Find the relative extrema of the following functions.
   a) $f(x) = 2x^2 + 3x - 4$
   b) $f(x) = x^3 + 3x^2 - 1$
   c) $f(x) = x^3 - 2x^2 - 4x + 4$
5. A pharmaceutical company finds that the total cost $C(x)$ (in dollars) of manufacturing $x$ cartons of particular drug per day is given by $C(x) = 400 + 4x + 0.0001x^2$. Each carton can be sold at a price of $p$ dollars, where $p$ is related to $x$ by the demand equation $p = 10 - 0.0004x$. If all cartons that are manufactured can be sold, find the daily level of production that will yield a maximum profit for the company.