Lecture 17

Learning Objectives

At the end of this class, students should be able to:

- apply differentiation rules

Derivative of Logarithmic Function

The derivative of $\ln x$ is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$

and $\frac{d}{dx}[\ln h(x)] = \frac{1}{h(x)}\left[\frac{d}{dx}\{h(x)\}\right]$, where h is a differentiable function of x.

For example, if we have $y = x\ln x$, then

$$\begin{aligned} y' &= x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x) = x \times \frac{1}{x} + \ln x \times 1 \\ &= 1 + \ln x . \end{aligned}$$

Illustration

Differentiate the function $y = \ln(x^2 + 5)^4$.

Solution

We have $y = \ln(x^2 + 5)^4$, which can be written as $y = 4\ln(x^2 + 5)$. Then

$$\begin{aligned} y' &= 4\frac{d}{dx}\{\ln(x^2 + 5)\} \\ &= 4 \times \frac{1}{(x^2 + 5)} \times \frac{d}{dx}(x^2 + 5) \\ &= \frac{4}{(x^2 + 5)} \times 2x \\ &= \frac{8x}{(x^2 + 5)} \end{aligned}$$

Note: $\frac{d}{dx}(\log_a x) = \frac{1}{x(\ln a)}, \ x > 0$

Derivative of Exponential Function

The derivative of e^x is

$$\frac{d}{dx}(e^x) = e^x$$

and $\frac{d}{dx}[e^{h(x)}] = e^{h(x)}\frac{d}{dx}\{h(x)\}$, where h is a differentiable function of x.

For example, if we have $y = e^{x^2+1}$, then $y' = e^{x^2+1}\frac{d}{dx}(x^2 + 1) = 2x e^{x^2+1}$.

Note: $\frac{d}{dx}(a^x) = a^x \ln a$

Illustration

Differentiate the function $y = (2e^x)^3$.

Solution

We have $y = (2e^x)^3$. It can be written as $y = 8e^{3x}$. Then

$$\begin{aligned}\frac{dy}{dx} &= 8\frac{d}{dx}(e^{3x}) \\ &= 8e^{3x} \times \frac{d}{dx}(3x) \\ &= 8e^{3x} \times 3 \\ &= 24e^{3x}\end{aligned}$$

Illustration

If $f(x) = \frac{\ln x}{e^{x^2}}$, find $f'(x)$.

Solution

We have $f(x) = \frac{\ln x}{e^{x^2}}$. Then

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(\frac{\ln x}{e^{x^2}}\right) \\ &= \frac{e^{x^2}\frac{d}{dx}(\ln x) - \ln x \frac{d}{dx}\left(e^{x^2}\right)}{\left(e^{x^2}\right)^2} = \frac{e^{x^2} \times \frac{1}{x} - \ln x \times e^{x^2}\frac{d}{dx}\left(x^2\right)}{\left(e^{x^2}\right)^2} \\ &= \frac{\frac{e^{x^2}}{x} - 2xe^{x^2}\ln x}{\left(e^{x^2}\right)^2} = \frac{1 - 2x^2 \ln x}{xe^{x^2}}\end{aligned}$$

Logarithmic Differentiation

We find the derivative by taking logarithm of the function if its exponent also contains variables. Moreover, we can use logarithm if the function that has to be differentiated contains product or quotient of many function (factors).

Illustration

Find $\frac{dy}{dx}$ if $y = x^y$.

Solution

We have, $y = x^y$

Taking *ln* on both sides, we get

$$\ln y = y \ln x$$

Differentiating both sides with respect to *x*, we get

$$\frac{d}{dx}(\ln y) = y\frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(y)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \ln x \frac{dy}{dx}$$

$$\left(\frac{1}{y} - \ln x\right)\frac{dy}{dx} = \frac{y}{x}$$

$\therefore \quad \frac{dy}{dx} = \frac{y^2}{x(1-y\ln x)}$

Derivative of One Function with respect to another Function

Suppose $y_1 = f_1(x)$ and $y_2 = f_2(x)$ are two functions. Derivative of y_1 with respect to y_2 is denoted by $\frac{dy_1}{dy_2}$ and defined by

$$\frac{dy_1}{dy_2} = \frac{\left(\frac{dy_1}{dx}\right)}{\left(\frac{dy_2}{dx}\right)}$$

Illustration

Differentiate $\ln(x^2 + 2x + 1)$ with respect to x^2.

Solution

We know that

$$\frac{d}{dx^2}\{\ln(x^2 + 2x + 1)\} = \frac{\frac{d}{dx}\{\ln(x^2 + 2x + 1)\}}{\frac{d}{dx}(x^2)} = \frac{\frac{2x+2}{(x^2 + 2x + 1)}}{2x} = \frac{(x+1)}{x(x^2 + 2x + 1)}$$

Exercise for Reader

1. Differentiate the following functions.
   a) e^{x^2+2x-5}
   b) xe^{-x^2}
   c) $e^{\sqrt{3x}}$
   d) $\ln(x^2+3x-5)$
   e) $\ln\sqrt{x^2+5}$
   f) $e^{\ln x + 2\ln 3x}$
2. Find $\frac{dy}{dx}$ if $x = a\frac{1-t^2}{1+t^2}$ and $y = \frac{2bt}{1+t^2}$.
2. Find $\frac{dy}{dx}$ if:
   a) $y = x^x$
   b) $y = xe^x + \ln y$
3. Differentiate $x^4 + 3x^2 + 7$ with respect to x^2 at $x = 3$.