Lecture 16

Learning Objectives
At the end of this class, students should be able to:

- apply differentiation rules

Quotient Rule
If $f(x) = u(x)/v(x)$, where u and v are differentiable functions and $v(x) \neq 0$, then

$$f'(x) = \frac{v\,u' - u\,v'}{v^2}$$

For example, if $f(x) = \dfrac{x^2 - 5}{3 + x}$, then

$$\begin{aligned} f'(x) &= \frac{(3 + x)\times 2x - (x^2 - 5)\times 1}{(3 + x)^2} \\ &= \frac{6x + 2x^2 - x^2 + 5}{(3 + x)^2} \\ &= \frac{x^2 + 6x + 5}{(3 + x)^2} \end{aligned}$$

General Power Rule
If $f(x) = [u(x)]^n$, where u is a differentiable function and n is a real number, then

$$f'(x) = n[u(x)]^{n-1}.\,u'(x)$$

For example, if $f(x) = (5x^2 - 3)^{3/2}$, then

$$\begin{aligned} f'(x) &= (3/2)(5x^2 - 3)^{1/2}\frac{d}{dx}(5x^2 - 3) \\ &= (3/2)(5x^2 - 3)^{1/2} \times 10x \\ &= 15x(5x^2 - 3)^{1/2} \end{aligned}$$

Chain Rule (Differentiation of a Function of a Function)
Let us illustrate the idea of the situation with the help of the following example. Let x be a land, y be wheat and z be bread. Assume that, for every unit of x (land), we can produce 2 units of y (wheat), mathematically, we have $y = 2x$. Also, for every unit of y (wheat), we can produce 15 units of z (bread), and this is shown by $z = 15y$.

Thus, here we have, $z = f(y)$, and $y = g(x)$.

We have to answer the following question. What will be change in value of z (bread) when there is a one-unit change in the value of x (land)?

The amount of change of z due to a small unit change of x is given by

$$\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx}$$

Here, $\frac{dz}{dx} = 15 \times 2 = 30$

Thus, if $z = f(y)$, and $y = g(x)$, then $\frac{dz}{dx} = \frac{dz}{dy} . \frac{dy}{dx}$, which is called the chain rule.

Implicit Differentiation

A function that is expressed in the form $y = f(x)$, say, $y = 3x^2 - 7x + 2$ is called an explicit function, because the variable *y* is explicitly expressed as a function of *x*. A function which is expressed in the form $f(x, y) = 0$, say, $x^2 + 2xy + y^3 = 0$ is called an implicit function.

If the equation $f(x, y) = 0$ can be solved for *y*, we can explicitly write out the function $y = f(x)$, and find its derivatives by the methods learned before. But, if the equation $f(x, y) = 0$ cannot be solved for *y* explicitly, we have a simple technique based on the chain rule that can be used to find $\frac{dy}{dx}$. Thus, the process of finding value of $\frac{dy}{dx}$ without solving the equation for *y* is called implicit differentiation. It consists of differentiating both sides of the given equation with respect to x and then solving algebraically for $\frac{dy}{dx}$. The following example illustrates the technique.

Illustration

Find $\frac{dy}{dx}$ of the function: $x^2 + 5xy + y^3 = 0$.

Solution

We have $x^2 + 5xy + y^3 = 0$.

Differentiating both sides with respect to *x*, we get

$$\frac{d}{dx}(x^2) + 5\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = 0$$

or $$2x + 5\left\{x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right\} + \left\{\frac{d}{dy}(y^3)\right\}\frac{dy}{dx} = 0$$

or $$2x + 5\left\{x \times \frac{dy}{dx} + y \times 1\right\} + 3y^2\frac{dy}{dx} = 0$$

or $$2x + 5x\frac{dy}{dx} + 5y + 3y^2\frac{dy}{dx} = 0$$

or $$(5x + 3y^2)\frac{dy}{dx} = -(2x + 5y)$$

$$\therefore \frac{dy}{dx} = -\frac{(2x + 5y)}{(5x + 3y^2)}$$

Derivative of Parametric Functions

Parametric functions are those in which both the variables *x* and *y* are expressed in terms of a third variable called, the parameter. Let $x = f(t)$ and $y = g(t)$ be any two functions of *x* and *y* (*t* being the parameter). In order to find $\frac{dy}{dx}$, we differentiate these two functions separately with respect to *t* and then we apply the following relation:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \qquad \text{provided } \frac{dx}{dt} \neq 0$$

Illustration

Find $\frac{dy}{dx}$ if $x = 2a(t^3 + 5)$ and $y = 5a^2(t^2 + 7)$.

Solution

We have $x = 2a(t^3 + 5)$ and $y = 5a^2(t^2 + 7)$, then

$$\frac{dx}{dt} = 6at^2 \quad \text{and} \quad \frac{dy}{dt} = 10a^2 t$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{10a^2 t}{6at^2} = \frac{5a}{3t}$$

Exercise for Reader

1. Find $\frac{dy}{dx}$ if:
   a) $y = (4x^2 - 3x)(x^3 - 8x^2 + 12)$
   b) $y = \frac{x^2 + 1}{x^2 - 1}$
   c) $y = \sqrt{x^2 + 3}$
   d) $y = (2x - 5)^{5/4} (3x + 7)^{2/3}$
   e) $y = \sqrt{\frac{2x + 3}{x + 5}}$

2. Find $\frac{dy}{dx}$ if:
   a) $y = 5u^2 - 7u$, $u = x^2 + 3$
   b) $y = 3u^2 - 6u + 2$, $u = v^2 - 1$; $v = 2x$

3. Find the value of $\frac{dy}{dx}$ if:
   a) $y = x^2 y^3 + x^3 y^2$
   b) $x^2 y + 2y^3 = 3x + 2y$