Lecture 15

Learning Objectives

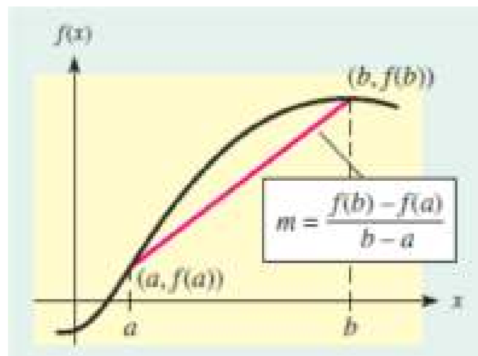
At the end of this class, students should be able to:

- identify average rate of change
- identify instantaneous rate of change
- apply differentiation rules

15.1 Average Rate of Change

The average rate of change of a function $y = f(x)$ from $x = a$ to $x = b$ is defined by

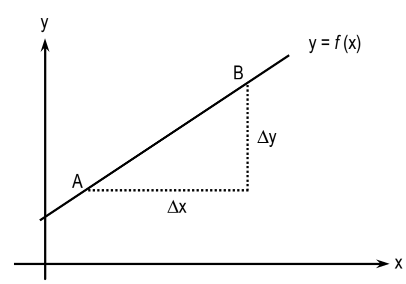
Average rate of change $= \frac{f(b)-f(a)}{b-a}$



The above figure shows that this average rate is the same as the slope of the segment joining the points $(a, f(a))$ and $(b, f(b))$.

Thus, if $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on a graph of a linear function, then the slope of the line segment can be found by applying the two-point formula.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



With linear function the slope is constant over the domain of the function. The slope provides an idea of the measure of the rate of change in the value of y with respect to a change in the value of x.

Illustration

The following table indicates annual sales (in dollars) for a Pharmaceutical Company during a selected period. At what average rate did annual sales increases between 1998 and 2000?

Year	1998	1999	2000	2001
Annual Sales (millions)	\$102.8	\$106.4	\$110.8	\$116.4

Solution

We know that the average rate of change $= \frac{\text{Change in sales}}{\text{Change in time}}$

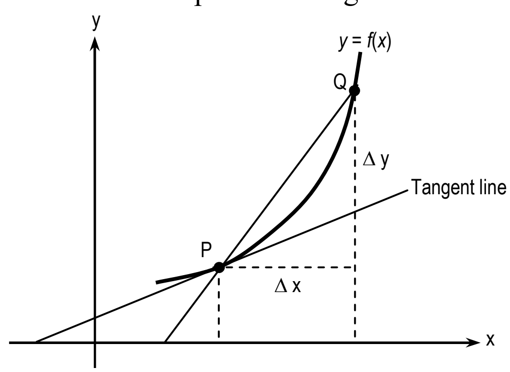
$= \frac{110.8-102.8}{2000-1998} = \frac{8}{2}$

= \$4 million per year

With nonlinear function the rate of change in the value of y with respect to a change in x is not constant. In the case of nonlinear functions, we try to find the slope of a function at a particular point in the domain.

15.2 Instantaneous Rate of Change

The slope of a curve at $x = a$ is the slope of the tangent line at $x = a$.



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the curve. Then the slope of straight line PQ is given by

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As point Q moves towards P (i.e., Δx approaches 0) along the curve $y = f(x)$, the value of $\frac{\Delta y}{\Delta x}$ also changes and will approach a limit, if it exists. This limit provides the slope of the curve at point P.

This limit is also known as the ***derivative*** of the function $f(x)$ at the point P and written as follows:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{derivative} = \frac{dy}{dx}$$

The symbol $\frac{dy}{dx}$ is used to express the derivative. The other symbols that are used for derivative are $\frac{df(x)}{dx}$ or $f'(x)$ or y', or y_1 or simply by f'.

The interpretation y' is the change in the dependent variable y due to small unit change in independent variable x. This type of change is also known as *instantaneous rate of change*. The process of obtaining the derivative is called differentiation of the function $f(x)$.

15.3 Rules of Differentiation

We use the following rules for finding derivative.

Derivative of a Constant Function

Derivative of a constant function is always zero. If $f(x) = c$, where c is any constant, then

$f'(x) = 0$

For example, if $f(x) = 6$, then $f'(x) = 0$.

Power Rule

If $f(x) = x^n$, where n is a real number, then

$f'(x) = nx^{n-1}$

For example, if $f(x) = x^5$, then $f'(x) = 5x^4$.

Constant Multiple Rule

If $f(x) = c.\ g(x)$, where c is any constant and g is a differentiable function, then

$f'(x) = c.\ g'(x)$

For example, if $f(x) = 5x^2$, then $f'(x) = 5\ (2x) = 10x$.

Sum or Difference Rule

If $f(x) = u(x) \pm v(x)$, where u and v are differentiable functions, then

$$f'(x) = u'(x) \pm v'(x)$$

For example, if $f(x) = 6x^2 - 5x + 7$, then $f'(x) = 12x - 5$.

Product Rule

If $f(x) = u(x).v(x)$, where u and v are differentiable functions, then

$f'(x) = u(x)\ v'(x) + v(x)u'(x)$

For example, if $f(x) = (4x^2 - 5x)(3 - 2x)$, then

$$\begin{aligned} f'(x) &= (4x^2 - 5x)\frac{d}{dx}(3 - 2x) + (3 - 2x)\frac{d}{dx}(4x^2 - 5x) \\ &= (4x^2 - 5x)(-2) + (3 - 2x)\ (8x - 5) \\ &= -8x^2 + 10x + 24x - 16x^2 - 15 + 10x \\ &= -24x^2 + 44x - 15 \end{aligned}$$

Exercise for Reader

1. Determine average rate of change in the value of y in moving from $x = 1$ to $x = 3$, where $y = x^2 + 2x - 4$.
2. Find the derivative of the following functions with respect to x.
   a) 7 b) $2x - 100$
   c) $5 - 7x$ d) $x^3 + 5x^2 + \frac{x}{12} + 8$
   e) $5/x$ f) $x^2 + 8x + \frac{1}{x} + \frac{7}{x^2} - \frac{4}{5x^3}$
   g) $\sqrt{x} + \frac{3}{\sqrt{2x}}$ h) $5x^{7/8} - 3\sqrt{x} + 1$
   i) $-\frac{5}{x^2} + x^{3/2} + \frac{1}{2\sqrt{x}} + \frac{x^2}{4} + 8$
   j) $(x^2 + 5)(2 - 7x)$
3. Find the slope of the graph of the function $f(x) = x^2 + 2x + 7$ at $(0, 2)$.