Lecture 14

Learning Objectives

At the end of this class, students should be able to:

- use various techniques for finding limit
- check the continuity of a function at a particular point
- identify the point(s) of discontinuity

14.1 Techniques of Finding Limits

Let $f(x)$ be a given function. To evaluate $\lim_{x \to c} f(x)$, we use the following techniques:

1. a) Substitute c for x in $f(x)$ to find $f(c)$.
   b) If $f(c) = k$ (a finite number), then the required limit is $f(c)$.
   c) If $f(c) = \frac{k}{0}$, the limit does not exist.
2. When $f(c)$ is not defined and takes the form: $\frac{0}{0}$.
   a) If possible, simplify $f(x)$ by factorization or rationalization so that we get the common factor $(x - c)$ in both numerator and denominator.
   b) As $(x - c) \neq 0$, so cancel out this common factor from numerator and denominator.
   c) Again, find $f_1(c)$ where $f_1(x)$ is the expression which remains after canceling $(x - c)$ from numerator and denominator.
   d) If $f_1(c) = k$ (a finite number), then the required limit is $f_1(c)$.
3. If $\lim_{x \to c} f(x)$ is not defined and takes the form: $\frac{\infty}{\infty}$.
   Divide the numerator and denominator of $f(x)$ by the highest power of x that has appeared either in the numerator or in the denominator of $f(x)$, then apply the limit.

Illustration

Evaluate $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$

Solution

The function $\frac{\sqrt{x+1}-1}{x}$ is not defined when $x = 0$, since $\frac{\sqrt{x+1}-1}{x} = \frac{0}{0}$.

Now,

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$$

$$= \lim_{x \to 0} \frac{x+1-1}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{x+1}+1)}$$

$$= \frac{1}{(\sqrt{0+1}+1)}$$

$$= \frac{1}{(1+1)}$$

$$= \frac{1}{2}$$

Illustration

Evaluate $\lim_{x \to \infty} \frac{5x}{3x^2 + 2x}$

Solution

$$\lim_{x \to \infty} \frac{5x}{3x^2 + 2x} = \lim_{x \to \infty} \frac{\frac{5x}{x^2}}{\frac{3x^2}{x^2} + \frac{2x}{x^2}}$$ [Numerator and denominator are divided by x^2]

$$= \lim_{x \to \infty} \frac{\frac{5}{x}}{3 + \frac{2}{x}}$$

$$= \frac{\frac{5}{\infty}}{3 + \frac{2}{\infty}}$$

$$= \frac{0}{3+0}$$

$$= 0$$

14.2 Continuity at a Point

A function is said to be continuous at point $x = a$ if

- $\lim_{x \to a} f(x)$ exists,
- the function is defined at $x = a$, i.e., $f(a)$ exists, and
- $\lim_{x \to a} f(x) = f(a)$

Illustration

Test the continuity of the function $f(x) = 3x - 5$ at $x = 2$.

Solution

We know that $\lim_{x \to 2} (3x - 5) = 1$, and $f(2) = 3 \times 2 - 5 = 1$.

Thus, $\lim_{x \to 2} (3x - 5) = f(2) = 1$. Hence given function is continuous at the specified point.

Note: A function f is continuous over an interval $[a, b]$ if it is continuous at every point within the interval. It is said to be discontinuous at $x = c$ if it is not continuous at $x = c$.

Polynomial functions are continuous in the entire domain, since these functions are defined for every real number and $\lim_{x \to a} = f(a)$ for all real a.

Illustration

Discuss the continuity of the function at the specified point.

$$f(x) = \begin{cases} 2x - 1 & \text{for } 0 \le x \le 1 \\ x & \text{for } x > 1 \end{cases} \quad \text{at } x = 1$$

Solution

Here, left hand limit is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2x - 1) = 2 \times 1 - 1 = 1$$

Right hand limit is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} x = 1$$

and value of the function at $x = 1$ is

$$f(1) = 2 \times 1 - 1 = 1$$

Thus $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

Illustration

Discuss the continuity of the function at the specified point.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\ 3 & \text{for } x = 2 \end{cases} \quad \text{at } x = 2$$

Solution

Here,
$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{(x - 2)} \\ &= \lim_{x \to 2} (x + 2) = 2 + 2 = 4 \end{aligned}$$

and $f(2) = 3$

We see that, $\lim_{x \to 2} f(x) \neq f(2)$.

Hence, the given function is not continuous at $x = 2$.

Illustration

What are the discontinuities, if any, of the function: $f(x) = \frac{x^2 - 5}{x^3 + x^2 - x - 1}$?

Solution

The given function is not defined when the denominator becomes zero,

i.e. $x^3 + x^2 - x - 1 = 0$

or $x^2(x + 1) - (x + 1) = 0/$

or $(x^2 - 1)(x + 1) = 0$

or $(x - 1)(x + 1)(x + 1) = 0$

Thus, the denominator becomes zero for $x = -1$ or $x = 1$. Since the given function is not defined when $x = 1$ or $x = -1$, the function is discontinuous at these two points.

Exercise for Reader

1. Evaluate the following limits.

a) $\lim_{x \to -2} \frac{4-x^2}{2x^2+x^3}$

b) $\lim_{x \to 1} \frac{x-1}{x^3+x^2-2x}$

c) $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1}$

d) $\lim_{x \to 1} \frac{x}{x-1}$

e) $\lim_{x \to \infty} \frac{2x+7}{x-1}$

f) $\lim_{x \to \infty} \frac{2x^3+8x+10}{x^3-x}$

2. Discuss the continuity of the following function at $x = 3$.

$$f(x) = \begin{cases} 2x+3 & \text{for} \quad x > 3 \\ 3x & \text{for} \quad x < 3 \\ 9 & \text{for} \quad x = 3 \end{cases}$$

3. Find the values of A and B if the following function is continuous in the interval $[-1, 2]$

$$f(x) = \begin{cases} x+1 & \text{for} \quad -1 \le x \le 0 \\ Ax+B & \text{for} \quad 0 < x < 1 \\ -1 & \text{for} \quad 1 \le x \le 2 \end{cases}$$

4. In the following exercises, determine whether there are any discontinuities and, if so, where they occur? Find the intervals on which the function is continuous.

a) $f(x) = \frac{x^2-9}{x+3}$

b) $f(x) = \frac{2}{x^3-x}$

c) $f(x) = \begin{cases} 1-x & \text{if } x \le 2 \\ x-1 & \text{if } x > 2 \end{cases}$